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CHAPTER

2

SETS, FUNCTIONS AND GROUPS

SET

A well defined collection of distinct objects is named as a set.

The objects in a set are called its members or elements.

There are three different ways of describing a set.

- (i) The descriptive method.
- (ii) The tabular method.
- (iii) Set builder method.

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EXERCISE 2.1

Q.1 Write the following sets in set builder notation.

- (i) $\{1, 2, 3, \dots, 1000\}$ (ii) $\{0, 1, 2, \dots, 100\}$
- (iii) $\{0, \pm 1, \pm 2, \dots, \pm 1000\}$ (iv) $\{0, -1, -2, \dots, -500\}$
- (v) $\{100, 101, 102, \dots, 400\}$
- (vi) $\{-100, -101, -102, \dots, -500\}$
- (vii) $\{\text{Peshawar, Lahore, Karachi, Quetta}\}$
- (viii) $\{\text{January, June, July}\}$
- (ix) The set of all odd natural numbers
- (x) The set of all rational number
- (xi) The set of all real number between 1 and 2
- (xii) The set of all integers between -100 and 1000

Solution:

- (i) $\{1, 2, 3, \dots, 1000\}$

Set builder notation of given set is $\{x \mid x \in \mathbb{N} \wedge x \leq 1000\}$

- (ii) $\{0, 1, 2, \dots, 100\}$

Set builder notation of the given set is $\{x \mid x \in \mathbb{W} \wedge x \leq 100\}$

- (iii) $\{0, \pm 1, \pm 2, \dots, \pm 1000\}$

Set builder notation of the given set is $\{x \mid x \in \mathbb{Z} \wedge -1000 \leq x \leq 1000\}$



- (iv) $\{0, -1, -2, \dots - 500\}$
Set builder notation of the given set is $\{x \mid x \in \mathbb{Z} \wedge -500 \leq x \leq 0\}$
- (v) $\{100, 101, 102, \dots 400\}$
Set builder notation of the given set is $\{x \mid x \in \mathbb{N} \wedge 100 \leq x \leq 400\}$
- (vi) $\{-100, -101, -102, \dots - 500\}$
Set builder notation of the given set is $\{x \mid x \in \mathbb{Z} \wedge -500 \leq x \leq -100\}$
- (vii) $\{\text{Peshawar, Lahore, Karachi, Quetta}\}$
Set builder notation of the given set is $\{x \mid x \text{ is the capital of province of Pakistan}\}$
- (viii) $\{\text{January, June, July}\}$
Set builder notation of the given set is $\{x \mid x \text{ is the name of month starts with "J"}\}$
- (ix) The set of all odd natural numbers
Set builder notation of the given set is $\{x \mid x \text{ is an odd natural number}\}$
- (x) The set of all rational number
Set builder notation of the given set is $\{x \mid x \in \mathbb{Q}\}$
- (xi) The set of all real number between 1 and 2
Set builder notation of the given set is $\{x \mid x \in \mathbb{R} \wedge 1 < x < 2\}$
- (xii) The set of all integers between -100 and 1000
Set builder notation of the given set is $\{x \mid x \in \mathbb{Z} \wedge -100 < x < 1000\}$

Q.2 Write each of the following sets in descriptive and tabular form:

- | | |
|---|--|
| (i) $\{x \mid x \in \mathbb{N} \wedge x \leq 10\}$ | (ii) $\{x \mid x \in \mathbb{N} \wedge 4 < x < 12\}$ |
| (iii) $\{x \mid x \in \mathbb{Z} \wedge 5 < x < 5\}$ | (iv) $\{x \mid x \in \mathbb{E} \wedge 2 < x \leq 4\}$ |
| (v) $\{x \mid x \in \mathbb{Z} \wedge -5 < x < 5\}$ | (vi) $\{x \mid x \in \mathbb{O} \wedge 3 < x < 12\}$ |
| (vii) $\{x \mid x \in \mathbb{E} \wedge 4 \leq x \leq 10\}$ | (viii) $\{x \mid x \in \mathbb{E} \wedge 4 < x < 6\}$ |
| (ix) $\{x \mid x \in \mathbb{O} \wedge 5 \leq x \leq 7\}$ | (x) $\{x \mid x \in \mathbb{O} \wedge 5 \leq x < 7\}$ |
| (xi) $\{x \mid x \in \mathbb{N} \wedge x + 4 = 0\}$ | (xii) $\{x \mid x \in \mathbb{Q} \wedge x^2 = 2\}$ |
| (xiii) $\{x \mid x \in \mathbb{R} \wedge x = x\}$ | (xiv) $\{x \mid x \in \mathbb{Q} \wedge x = -x\}$ |
| (xv) $\{x \mid x \in \mathbb{R} \wedge x \neq 2\}$ | (xvi) $\{x \mid x \in \mathbb{R} \wedge x \notin \mathbb{Q}\}$ |

Solution:

- (i) $\{x \mid x \in \mathbb{N} \wedge x \leq 10\}$
Descriptive form is: The set of the first ten natural numbers.
The tabular form is: $\{1, 2, 3, \dots 10\}$
- (ii) $\{x \mid x \in \mathbb{N} \wedge 4 < x < 12\}$
Descriptive form is: The set of the natural numbers between 4 and 12.
Tabular form is: $\{5, 6, 7, \dots 11\}$



(iii) $\{x \mid x \in \mathbb{Z} \wedge -5 < x < 5\}$

Descriptive form is: The set of integers between -5 and 5 .

Tabular form is: $\{-4, -3, -2, \dots, 4\}$

(iv) $\{x \mid x \in \mathbb{E} \wedge 2 < x \leq 4\}$

Descriptive form is: The set of even integers between 2 and 5 .

Tabular form is: $\{4\}$

(v) $\{x \mid x \in \mathbb{P} \wedge x < 12\}$

Descriptive form is: The set of prime numbers less than 5 .

Tabular form is: $\{2, 3, 5, 7, 11\}$

(vi) $\{x \mid x \in \mathbb{O} \wedge 3 < x < 12\}$

Descriptive form is: The set of odd integers between 3 and 12 .

Tabular form is: $\{5, 7, 9, 11\}$

(vii) $\{x \mid x \in \mathbb{E} \wedge 4 \leq x \leq 10\}$

Descriptive form is: The set of even integers from 4 upto 10 .

Tabular form is: $\{4, 6, 8, 10\}$

(viii) $\{x \mid x \in \mathbb{E} \wedge 4 < x < 6\}$

Descriptive form is: The set of even integers between 4 and 6 .

Tabular form is: $\{ \}$

(ix) $\{x \mid x \in \mathbb{O} \wedge 5 \leq x \leq 7\}$

Descriptive form is: The set of odd integers from 5 upto 7 .

Tabular form is: $\{5, 7\}$

(x) $\{x \mid x \in \mathbb{O} \wedge 5 < x < 7\}$

Descriptive form is: The set of odd integers between 5 and 7 .

Tabular form is: $\{ \}$

(xi) $\{x \mid x \in \mathbb{N} \wedge x + 4 = 0\}$

Descriptive form is: The set of the natural numbers x satisfying $x + 4 = 0$.

Tabular form is: $\{ \}$

(xii) $\{x \mid x \in \mathbb{Q} \wedge x^2 = 2\}$

Descriptive form is: The set of rational numbers x satisfying $x^2 = 2$.



Tabular form is: $\{ \}$

(xiii) $\{x \mid x \in \mathbb{R} \wedge x = x\}$

Descriptive form is: The set of real numbers x satisfying $x = x$.

Tabular form is: The set of real numbers.

(xiv) $\{x \mid x \in \mathbb{Q} \wedge x = -x\}$

Descriptive form is: The set of rational numbers x satisfying $x = -x$.

Tabular form is: $\{0\}$

(xv) $\{x \mid x \in \mathbb{R} \wedge x \neq 2\}$

Descriptive form is: The set of real numbers x satisfying $x \neq 2$.

Tabular form is: $\mathbb{R} - \{2\}$

(xvi) $\{x \mid x \in \mathbb{R} \wedge x \notin \mathbb{Q}\}$

Descriptive form is: The set of real numbers x which are not rational.

Tabular form is: $\mathbb{Q}'$

FINITE AND INFINITE SETS

If the number of elements in a set are finite then it is called finite set and if the number of elements in a set are infinite then it is called an infinite set.

Q.3 Which of the following sets are finite and which of these are infinite.

- (i) The set of students of your class
- (ii) The set of all schools in Pakistan
- (iii) The set of natural numbers between 3 and 10
- (iv) The set of rational numbers between 3 and 10
- (v) The set of real numbers between 0
- (vi) The set of rationales between 0 and 1
- (vii) The set of whole numbers between 0 and 1
- (viii) The set of all leaves of trees in Pakistan
- (ix) $P(\mathbb{N})$
- (x) $P\{a, b, c\}$
- (xi) $\{1, 2, 3, \dots\}$
- (xii) $\{1, 2, 3, \dots, 100000000\}$
- (xiii) $\{x \mid x \in \mathbb{R} \wedge x \neq x\}$
- (xiv) $\{x \mid x \in \mathbb{R} \wedge x^2 = -16\}$
- (xv) $\{x \mid x \in \mathbb{Q} \wedge x^2 = 5\}$



(xvi) $\{x \mid x \in \mathbb{Q} \wedge 0 \leq x \leq 1\}$

Solution:

- (i) Finite set.
- (ii) Finite set.
- (iii) Finite set.
- (iv) Infinite set.
- (v) Infinite set.
- (vi) Infinite set.
- (vii) Finite set.
- (viii) Infinite set.
- (ix) Infinite set.
- (x) Finite set.
- (xi) Infinite set.
- (xii) Finite set.
- (xiii) Finite set.
- (xiv) Finite set.
- (xv) Finite set.
- (xvi) Infinite set.

SUBSET

If every element of a set A is an element of set B , then A is said to be a subset of B i.e. $A \subseteq B$.

PROPER SUBSET

If A is a subset of B and B contains at least one element which is not an element of A , then A is said to be a proper subset of B .

IMPROPER SUBSET

If A is subset of B and $A = B$, then we say that A is an improper subset of B .



Q.4 Write two proper subsets of each of the following sets

(i) $\{a, b, c\}$

(Gujranwala Board 2007)

(ii) $\{0, 1\}$

(Lahore Board 2006)

(iii) N

(iv) z

(v) Q

(vi) $\mathcal{R}$

(vii) W

(viii) $\{x \mid x \in Q \wedge 0 < x \leq 2\}$

Solution:

- (i) Two proper subsets are $\{a\}, \{b\}$
- (ii) Two proper subsets are $\{0\}, \{1\}$
- (iii) Two proper subsets are $\{10\}, \{4\}$
- (iv) Two proper subsets are $\{1\}, \{3\}$
- (v) Two proper subsets are $\{1\}, \{2\}$
- (vi) Two proper subsets are $\{0\}, \{1\}$
- (vii) Two proper subsets are $\{1\}, \{5\}$
- (viii) Two proper subsets are $\{1\}, \{1, 2\}$

Q.5 Is there any set which has no proper subset? If so name that set?

(Lahore Board 2009)

Solution:

Yes, there is a set which has no proper subset and it is ϕ or $\{ \}$ i.e. empty set.

Q.6 What is the difference between $\{a, b\}$ and $\{\{a, b\}\}$.

Solution:

$\{a, b\}$ contains two elements but $\{\{a, b\}\}$ contains only one element $\{a, b\}$.

Q.7 Which of the following sentences are true and which of them are false.

(i) $\{1, 2\} = \{2, 1\}$

(ii) $\phi \subseteq \{\{a\}\}$

(iii) $\{a\} \supseteq \{\{a\}\}$

(iv) $\{a\} \in \{\{a\}\}$

(v) $a \in \{\{a\}\}$

(vi) $\phi \in \{\{a\}\}$

Solution:

- (i) $\{1, 2\} = \{2, 1\}$ True
- (ii) $\phi \subseteq \{\{a\}\}$ True
- (iii) $\{a\} \subseteq \{\{a\}\}$ False
- (iv) $\{a\} \in \{\{a\}\}$ True
- (v) $a \in \{\{a\}\}$ False
- (vi) $\phi \in \{\{a\}\}$ False



POWER SET

The power set of a set S denoted by $P(S)$ is the set containing all the possible subsets of S .

If number of elements in $S = m$ then number of elements in $P(S) = 2^m$.

or

if $n(S) = m$

then $n P(S) = 2^m$.

Q.8 What is the number of elements of the power set of each of the following sets?

(i) $\{ \}$

(ii) $\{0, 1\}$

(iii) $\{1, 2, 3, 4, 5, 6, 7\}$

(iv) $\{0, 1, 2, 3, 4, 5, 6, 7\}$

(v) $\{a, \{b, c\}\}$

(vi) $\{\{a, b\}, \{b, c\}, \{d, e\}\}$

Solution:

(i) $\{ \}$

Let $S = \{ \} \Rightarrow n(s) = 0$

$\Rightarrow n P(s) = 2^0 = 1$

(ii) $\{0, 1\}$

Let $s = \{0, 1\} \Rightarrow n(s) = 2$

$\Rightarrow n P(s) = 2^2 = 4$

(iii) $\{1, 2, 3, 4, 5, 6, 7\}$

Let $s = \{1, 2, 3, 4, 5, 6, 7\} \Rightarrow n(s) = 7$

$\Rightarrow n P(s) = 2^7 = 128$

(iv) $\{0, 1, 2, 3, 4, 5, 6, 7\}$

Let $s = \{0, 1, 2, 3, 4, 5, 6, 7\} \Rightarrow n(s) = 8$

$\Rightarrow n P(s) = 2^8 = 256$

(v) $\{a, \{b, c\}\}$

Let $s = \{a, \{b, c\}\} \Rightarrow n(s) = 2$

$\Rightarrow n P(s) = 2^2 = 4$

(vi) $\{\{a, b\}, \{b, c\}, \{d, e\}\}$

Let $s = \{\{a, b\}, \{b, c\}, \{d, e\}\} \Rightarrow n(s) = 3$

$\Rightarrow n P(s) = 2^3 = 8$



Q.9 Write down the power set of each of the following sets:

(i) $\{9, 11\}$

(ii) $\{+, -, \times, \div\}$

(iii) $\{\phi\}$

(iv) $\{a, \{b, c\}\}$

Solution:

(i) $\{9, 11\}$

Let $S = \{9, 11\}$

$P(S) = \{\phi, \{9\}, \{11\}, \{9, 11\}\}$

(ii) $\{+, -, \times, \div\}$

Let $S = \{+, -, \times, \div\}$

$P(S) = \{\phi, \{+\}, \{-\}, \{\times\}, \{\div\}, \{+, -\}, \{+, \times\}, \{+, \div\}, \{-, \times\},$
 $\{-, \div\}, \{\times, \div\}, \{+, -, \times\}, \{+, -, \div\}, \{-, \times, \div\},$
 $\{+, \times, \div\}, \{+, -, \times, \div\}\}$

(iii) $\{\phi\}$

Let $S = \{\phi\}$

$\Rightarrow P(S) = \{\phi, \{\phi\}\}$

(iv) $\{a, \{b, c\}\}$

Let $S = \{a, \{b, c\}\}$

$\Rightarrow P(S) = \{\phi, \{a\}, \{\{b, c\}\}, \{a, \{b, c\}\}\}$

EQUAL SETS

Two sets A and B are equal iff they have the same elements.

EQUIVALENT SETS

Two sets A and B are equivalent if a (1 – 1) correspondence can be established between A and B.

Q.10 Which pair of sets are equivalent? Which of them are also equal?

(i) $\{a, b, c\}, \{1, 2, 3\}$

(ii) The set of the first 10 whole numbers, $\{0, 1, 2, 3, \dots, 9\}$

(iii) Set of angles of a quadrilateral ABCD, set of the sides of the same quadrilateral

(iv) Set of the sides of a hexagon ABCDEF, Set of the angles of the same hexagon

(v) $\{1, 2, 3, 4, \dots\}, \{2, 4, 6, 8, \dots\}$

(vi) $\{1, 2, 3, 4, \dots\}, \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$

(vii) $\{5, 10, 15, 20, \dots, 55555\}, \{5, 10, 15, 20, \dots\}$



Solution:

- (i) $\{a, b, c\}, \{1, 2, 3\}$

Given sets are equivalent.

- (ii) The set of the first 10 whole members, $\{0, 1, 2, 3, \dots, 9\}$

The given sets are equivalent and also equal.

- (iii) Set of angles of a quadrilateral ABCD set of the sides of the same quadrilateral

The given sets are equivalent.

- (iv) Set of the sides of a hexagon ABCDEF, Set of the angles of the same hexagon

The given sets are equivalent.

- (v) $\{1, 2, 3, 4, \dots\}, \{2, 4, 6, 8, \dots\}$

The given sets are equivalent.

- (vi) $\{1, 2, 3, 4, \dots\}, \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$

The given sets are equivalent.

- (vii) $\{5, 10, 15, 20, \dots, 55555\}, \{5, 10, 15, 20, \dots\}$

The given sets are not equivalent.



VENN DIAGRAMS

Venn diagrams are used to describe a relation among the sets. In these diagrams, a rectangular region represents the universal set and circular closed curves represent the subsets.

EXERCISE 2.2

Q.1 Exhibit $A \cup B$ and $A \cap B$ by Venn diagrams in the following cases.

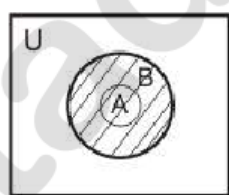
(i) $A \subseteq B$ (ii) $B \subseteq A$ (iii) $A \cup A'$

(iv) A and B are disjoint sets.

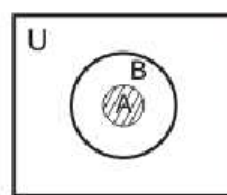
(v) A and B are overlapping sets.

Solution:

(i) $A \subseteq B$



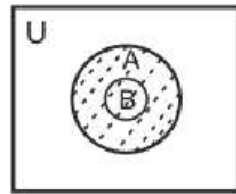
$A \cup B$



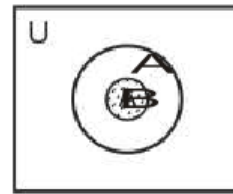
$A \cap B$



(ii) $B \subseteq A$

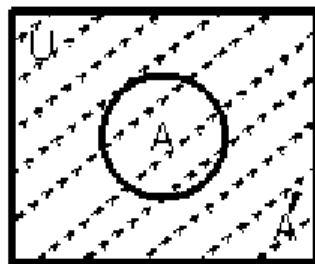


$A \cup B$

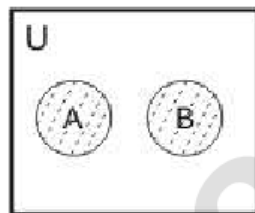


$A \cap B$

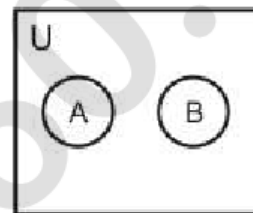
(iii) $A \cup A'$



(iv) A and B are disjoint

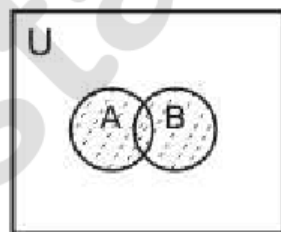


$A \cup B$

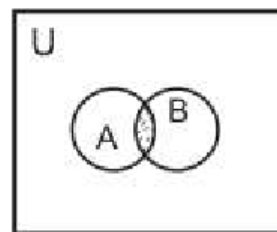


$A \cap B$

(v) A and B are overlapping sets.



$A \cup B$



$A \cap B$



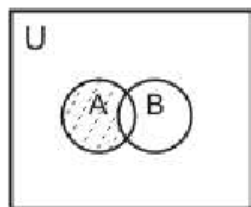
Q.2 Show $A - B$ and $B - A$ by Venn Diagram when

(i) A and B are overlapping sets

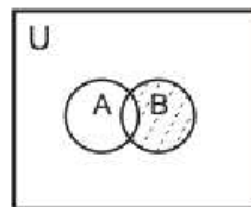
(ii) $A \subseteq B$ (iii) $B \subseteq A$

Solution:

(i) A and B are overlapping sets

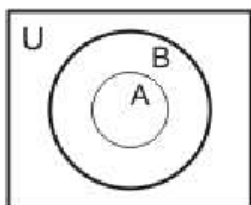


$A - B$

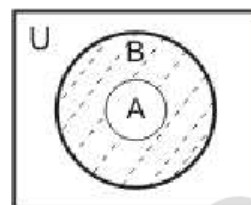


$B - A$

(ii) $A \subseteq B$

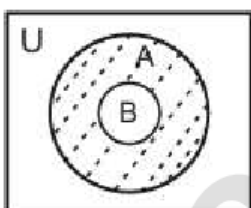


$A - B$

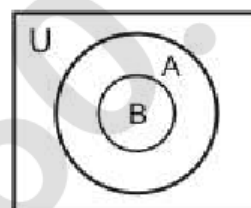


$B - A$

(iii) $B \subseteq A$



$A - B$



$B - A$

Q.3 Under what conditions on A and B are the following statements true?

(i) $A \cup B = A$ (ii) $A \cup B = B$ (iii) $A - \phi = \phi$

(iv) $A \cap B = B$ (v) $n(A \cup B) = n(A) + n(B)$

(vi) $n(A \cup B) = n(A)$ (vii) $A - B = A$

(viii) $n(A \cap B) = 0$ (ix) $A \cup B = U$

(x) $A \cup B = B \cup A$ (xi) $n(A \cap B) = n(B)$

(xii) $U - A = \phi$

Solution:

(i) $A \cup B = A$

if $B \subseteq A$

(ii) $A \cup B = B$

if $A \subseteq B$



- (iii) $A - B = A$
if $A \cap B = \phi$
- (iv) $A \cap B = B$
if $B \subseteq A$
- (v) $n(A \cup B) = n(A) + n(B)$
if $A \cap B = \phi$
- (vi) $n(A \cap B) = n(A)$
if $A \subseteq B$
- (vii) $A - B = A$
if $A \cap B = \phi$ or $B = \phi$
- (viii) $n(A \cap B) = 0$
if $A \cap B = \phi$
- (ix) $A \cup B = U$
if $A = B^c$
- (x) $A \cup B = B \cup A$
it holds always.
- (xi) $n(A \cap B) = n(B)$
if $B \subseteq A$
- (xii) $U - A = \phi$
if $A = U$

Q.4 Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$,
 $A = \{2, 4, 6, 8, 10\}$, $B = \{1, 2, 3, 4, 5\}$ and $C = \{1, 3, 5, 7, 9\}$

List the members of the following sets

- (i) A^c (ii) B^c (iii) $A \cup B$ (iv) $A - B$
 (v) $A \cap C$ (vi) $A^c \cup C^c$ (vii) $A^c \cup C$ (viii) U^c

Solution:

- (i) A^c
 $A^c = U - A$
 $= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} - \{2, 4, 6, 8, 10\}$
 $= \{1, 3, 5, 7, 9\}$



(ii) B^c

$$\begin{aligned} B^c &= U - B \\ &= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} - \{1, 2, 3, 4, 5\} \\ &= \{6, 7, 8, 9, 10\} \end{aligned}$$

(iii) $A \cup B$

$$\begin{aligned} A \cup B &= \{2, 4, 6, 8, 10\} \cup \{1, 2, 3, 4, 5\} \\ &= \{1, 2, 3, 4, 5, 6, 8, 10\} \end{aligned}$$

(iv) $A - B$

$$\begin{aligned} A - B &= \{2, 4, 6, 8, 10\} - \{1, 2, 3, 4, 5\} \\ &= \{6, 8, 10\} \end{aligned}$$

(v) $A \cap C$

$$\begin{aligned} A \cap C &= \{2, 4, 6, 8, 10\} \cap \{1, 3, 5, 7, 9\} \\ &= \{ \} \end{aligned}$$

(vi) $A^c \cup C^c$

$$A^c \cup C^c = (U - A) \cup (U - C) \quad \dots\dots\dots (1)$$

$$U - A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} - \{2, 4, 6, 8, 10\} = \{1, 3, 5, 7, 9\}$$

$$U - C = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} - \{1, 3, 5, 7, 9\} = \{2, 4, 6, 8, 10\}$$

Put in equation (1)

$$\begin{aligned} A^c \cup C^c &= \{1, 3, 5, 7, 9\} \cup \{2, 4, 6, 8, 10\} \\ &= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \end{aligned}$$

(vii) $A^c \cup C$

$$A^c \cup C = (U - A) \cup C \quad \dots\dots\dots (1)$$

$$U - A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} - \{2, 4, 6, 8, 10\} = \{1, 3, 5, 7, 9\}$$

Put in equation (1)

$$A^c \cup C = \{1, 3, 5, 7, 9\} \cup \{1, 3, 5, 7, 9\} = \{1, 3, 5, 7, 9\}$$

(viii) U^c

$$\begin{aligned} U^c &= U - U \\ &= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} - \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \\ &= \{ \} \end{aligned}$$



Q.5 Using the Venn diagrams, if necessary, find the single set equal to the following.

- (i) A^c (ii) $A \cap U$ (iii) $A \cup U$ (iv) $A \cup \phi$
 (v) $\phi \cap \phi$

Solution:

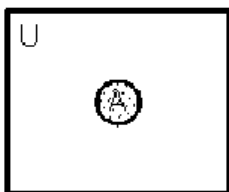
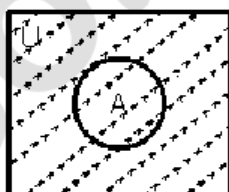
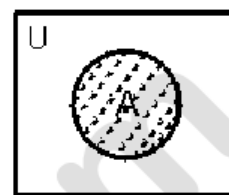
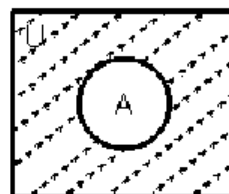
(i) A^c
 $A^c = U - A$

(ii) $A \cap U$

(iii) $A \cup U$

(iv) $A \cup \phi$

(v) $\phi \cap \phi$

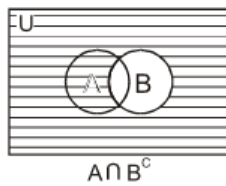
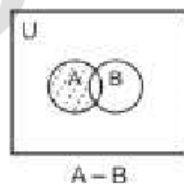


Q.6 Use Venn diagram to verify the following:

- (i) $A - B = A \cap B^c$ (ii) $(A - B)^c \cap B = B$

Solution:

(i) $A - B = A \cap B^c$

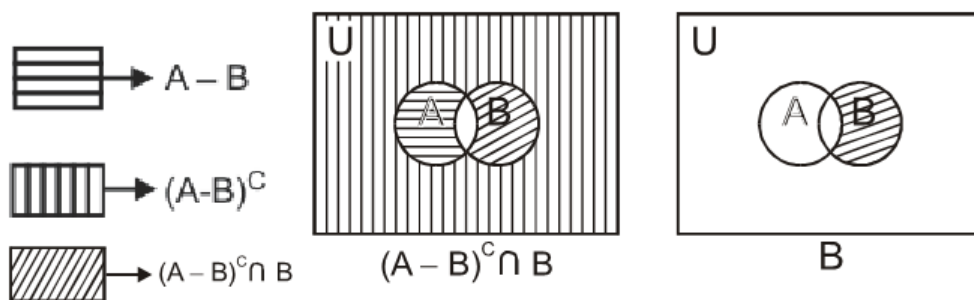


As the shaded portion in above two figures is same.

$\Rightarrow$ L.H.S. = R.H.S. $\Rightarrow A - B = A \cap B^c$



$$(ii) \quad (A - B)^c \cap B = B$$



As the shaded portion in above two figures is same $\Rightarrow (A - B)^c \cap B = B$

PROPERTIES OF UNION AND INTERSECTION

- | | | |
|-------|--|---|
| (i) | $A \cup B = B \cup A$ | Commutative property of union |
| (ii) | $A \cap B = B \cap A$ | Commutative property of intersection |
| (iii) | $A \cup (B \cap C) = (A \cup B) \cap C$ | Associative property of union |
| (iv) | $A \cap (B \cup C) = (A \cap B) \cup C$ | Associative property of intersection |
| (v) | $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ | Distributivity of union over intersection |
| (vi) | $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ | Distributivity of intersection over union |
| (vii) | $(A \cup B)' = A' \cap B'$ | |

De Morgan's Laws

$$(viii) \quad (A \cap B)' = A' \cup B'$$

PROOFS OF DE MORGAN'S LAWS AND DISTRIBUTIVE LAWS

$$(i) \quad (A \cup B)' = A' \cap B'$$

Let $x \in (A \cup B)'$

$$\Rightarrow x \notin A \cup B$$

$$\Rightarrow x \notin A \text{ and } x \notin B$$

$$\Rightarrow x \in A' \text{ and } x \in B'$$

$$\Rightarrow x \in A' \cap B'$$

$$\Rightarrow (A \cup B)' \subseteq A' \cap B' \quad \dots\dots\dots (1)$$

Now suppose that

$$x \in A' \cap B'$$

$$\Rightarrow x \in A' \text{ and } x \in B'$$

$$\Rightarrow x \notin A \text{ and } x \notin B$$

$$\Rightarrow x \notin A \cup B$$

$$\Rightarrow x \in (A \cup B)'$$

$$\Rightarrow A' \cap B' \subseteq (A \cup B)' \quad \dots\dots\dots (2)$$

From (1) and (2), we conclude that $(A \cup B)' = A' \cap B'$



$$(ii) \quad (A \cap B)' = A' \cup B'$$

Let $x \in (A \cap B)'$

$$\Rightarrow x \notin A \cap B$$

$$\Rightarrow x \notin A \text{ and } x \notin B$$

$$\Rightarrow x \in A' \text{ and } x \in B'$$

$$\Rightarrow x \in A' \cup B'$$

$$\Rightarrow (A \cap B)' \subseteq A' \cup B' \quad \dots\dots\dots (1)$$

Now suppose that

$$x \in A' \cup B'$$

$$\Rightarrow x \in A' \text{ or } x \in B'$$

$$\Rightarrow x \notin A \text{ or } x \notin B$$

$$\Rightarrow x \notin A \cap B$$

$$\Rightarrow x \in (A \cap B)'$$

$$\Rightarrow A' \cup B' \subseteq (A \cap B)' \quad \dots\dots\dots (2)$$

From (1) and (2), it is verified that $(A \cap B)' = A' \cup B'$

$$(iii) \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Let $x \in A \cup (B \cap C)$

$$\Rightarrow x \in A \text{ or } x \in B \cap C$$

$$\Rightarrow \text{if } x \in A$$

$$\text{then } x \in A \cup B \text{ and } x \in A \cup C$$

$$\text{and if } x \in B \cap C$$

$$\Rightarrow x \in B \text{ and } x \in C$$

$$\Rightarrow x \in A \cup B \text{ and } x \in A \cup C$$

$$\Rightarrow x \in (A \cup B) \cap (A \cup C)$$

$$\Rightarrow A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C) \quad \dots\dots\dots (1)$$

Now suppose that

$$x \in (A \cup B) \cap (A \cup C)$$

$$\Rightarrow x \in A \cup B \text{ and } x \in A \cup C$$

Now there are two cases either $x \in A$ or $x \notin A$

$$\text{if } x \in A \text{ then } x \in A \cup (B \cap C)$$

$$\text{if } x \notin A \text{ then } x \in B \text{ and } x \in C$$



$$\Rightarrow x \in B \cap C$$

$$\Rightarrow x \in A \cup (B \cap C)$$

$$\Rightarrow \text{In both cases } x \in A \cup (B \cap C) \quad \dots\dots\dots (2)$$

From (1) and (2)

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Hence proved.

$$(iv) \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Let $x \in A \cap (B \cup C)$

$$\Rightarrow x \in A \quad \text{and} \quad x \in B \cup C$$

$$\Rightarrow \text{if } x \in A \quad \text{and} \quad x \in B$$

$$\Rightarrow x \in A \cap B \Rightarrow x \in (A \cap B) \cup (A \cap C)$$

If $x \in A \quad \text{and} \quad x \in C$

$$\Rightarrow x \in A \cap C \Rightarrow x \in (A \cap B) \cup (A \cap C)$$

$$\Rightarrow A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C) \quad \dots\dots\dots (1)$$

Now suppose that

$$x \in (A \cap B) \cup (A \cap C)$$

$$\Rightarrow x \in A \cap B \quad \text{or} \quad x \in A \cap C$$

There are two cases.

Case I if $x \in A \cap B$

$$\Rightarrow x \in A \quad \text{and} \quad x \in B$$

$$\Rightarrow x \in A \quad \text{and} \quad x \in B \cup C$$

$$\Rightarrow x \in A \cap (B \cup C)$$

Case II if $x \in A \cap C$

$$\Rightarrow x \in A \quad \text{and} \quad x \in C$$

$$\Rightarrow x \in A \quad \text{and} \quad x \in B \cup C$$

$$\Rightarrow x \in A \cap (B \cup C)$$

$\Rightarrow$ In both cases

$$x \in A \cap (B \cup C)$$

$$\Rightarrow (A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C) \quad \dots\dots\dots (2)$$

From (1) and (2) we have

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Hence proved.



EXERCISE 2.3

Q.1 Verify the commutative properties of union and intersection for the following pair of sets.

(i) $A = \{1, 2, 3, 4, 5\}, \quad B = \{4, 6, 8, 10\}$

(ii) N, Z

(iii) $A = \{x \mid x \in \mathbb{R} \wedge x \geq 0\} \quad B = \mathbb{R}$

Solution:

(i) $A = \{1, 2, 3, 4, 5\} \quad B = \{4, 6, 8, 10\}$

$$A \cup B = \{1, 2, 3, 4, 5\} \cup \{4, 6, 8, 10\}$$

$$= \{1, 2, 3, 4, 5, 6, 8, 10\}$$

$$B \cup A = \{4, 6, 8, 10\} \cup \{1, 2, 3, 4, 5\}$$

$$= \{1, 2, 3, 4, 5, 6, 8, 10\}$$

$$\therefore A \cup B = B \cup A$$

$$A \cap B = \{1, 2, 3, 4, 5\} \cap \{4, 6, 8, 10\}$$

$$= \{4\}$$

$$B \cap A = \{4, 6, 8, 10\} \cap \{1, 2, 3, 4, 5\}$$

$$= \{4\}$$

$$\therefore A \cap B = B \cap A$$

Commutative properties of union and intersection are verified.

(ii) N, Z

$$N \cup Z = Z$$

and $Z \cup N = Z$

$$\Rightarrow N \cup Z = Z \cup N$$

Now

$$N \cap Z = N \quad \text{and} \quad Z \cap N = N$$

$$\Rightarrow N \cap Z = Z \cap N$$

Commutative properties of union and intersection are verified.

(iii) $A = \{x \mid x \in \mathbb{R} \wedge x \geq 0\} \quad B = \mathbb{R}$

Commutative property of union

$$A \cup B = \{x \mid x \in \mathbb{R} \wedge x \geq 0\} \cup \mathbb{R}$$

$$= \mathbb{R}$$



$$\begin{aligned}(B \cup A) &= R \cup \{x \mid x \in R \wedge x \geq 0\} \\ &= R\end{aligned}$$

$$\begin{aligned}A \cap B &= \{x \mid x \in R \wedge x \geq 0\} \cap R \\ &= \{x \mid x \in R \wedge x \geq 0\}\end{aligned}$$

$$\begin{aligned}B \cap A &= R \cap \{x \mid x \in R \wedge x \geq 0\} \\ &= \{x \mid x \in R \wedge x \geq 0\}\end{aligned}$$

$\Rightarrow$ Commutative properties of union and intersection are verified.

Q.2 Verify the properties for the sets A, B, C given below:

(i) **Associativity of Union** $A \cup (B \cup C) = (A \cup B) \cup C$

(a) $A = \{1, 2, 3, 4\}$ $B = \{3, 4, 5, 6, 7, 8\}$ $C = \{5, 6, 7, 9, 10\}$

(b) $A = \phi$, $B = \{0\}$, $C = \{0, 1, 2\}$

(c) N, Z, Q

(ii) **Associativity of intersection** $A \cap (B \cap C) = (A \cap B) \cap C$

(a) $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6, 7, 8\}$, $C = \{5, 6, 7, 9, 10\}$

(b) $A = \phi$, $B = \{0\}$, $C = \{0, 1, 2\}$

(c) N, Z, Q

(iii) **Distributivity of union over intersection** (Lahore Board 2005)

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

(a) $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6, 7, 8\}$, $C = \{5, 6, 7, 9, 10\}$

(b) $A = \phi$, $B = \{0\}$, $C = \{0, 1, 2\}$

(c) N, Z, Q

(iv) **Distributivity of intersection over union**

$$A \cup C (B \cap C) = (A \cup B) \cap (A \cup C)$$

(a) $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6, 7, 8\}$, $C = \{5, 6, 7, 9, 10\}$

(b) $A = \phi$ $B = \{0\}$ $C = \{0, 1, 2\}$, (c) N, Z, Q

Solution:

(i) **Associativity of union**

(a) $A = \{1, 2, 3, 4\}$ $B = \{3, 4, 5, 6, 7, 8\}$ $C = \{5, 6, 7, 9, 10\}$

i.e. $(A \cup B) \cup C = A \cup (B \cup C)$



L.H.S.

$$\begin{aligned}(A \cup B) \cup C &= (\{1, 2, 3, 4\} \cup \{3, 4, 5, 6, 7, 8\}) \cup \{5, 6, 7, 9, 10\} \\ &= \{1, 2, 3, 4, 5, 6, 7, 8\} \cup \{5, 6, 7, 9, 10\} \\ &= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}\end{aligned}$$

$$\begin{aligned}\text{R.H.S} &= A \cup (B \cup C) \\ &= \{1, 2, 3, 4\} \cup (\{3, 4, 5, 6, 7, 8\} \cup \{5, 6, 7, 9, 10\}) \\ &= \{1, 2, 3, 4\} \cup \{3, 4, 5, 6, 7, 8, 9, 10\} \\ &= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}\end{aligned}$$

$$\Rightarrow (A \cup B) \cup C = A \cup (B \cup C)$$

(b) $A = \phi$, $B = \{0\}$, $C = \{0, 1, 2\}$

To show $(A \cup B) \cup C = A \cup (B \cup C)$

L.H.S.

$$\begin{aligned}(A \cup B) \cup C &= (\phi \cup \{0\}) \cup \{0, 1, 2\} \\ &= \{0\} \cup \{0, 1, 2\} \\ &= \{0, 1, 2\}\end{aligned}$$

R.H.S.

$$\begin{aligned}A \cup (B \cup C) &= \phi \cup (\{0\} \cup \{0, 1, 2\}) \\ &= \phi \cup \{0, 1, 2\} \\ &= \{0, 1, 2\}\end{aligned}$$

$$\Rightarrow (A \cup B) \cup C = A \cup (B \cup C)$$

(c) N, Z, Q

To show $(N \cup Z) \cup Q = N \cup (Z \cup Q)$

$$(N \cup Z) \cup Q = Z \cup Q = Q$$

$$N \cup (Z \cup Q) = N \cup Q = Q$$

(ii) Associativity of intersection

(a) $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6, 7, 8\}$, $C = \{5, 6, 7, 9, 10\}$

Associativity of intersection is

$$(A \cap B) \cap C = A \cap (B \cap C)$$

L.H.S.

$$\begin{aligned}(A \cap B) \cap C &= (\{1, 2, 3, 4\} \cap \{3, 4, 5, 6, 7, 8\}) \cap \{5, 6, 7, 9, 10\} \\ &= \{3, 4\} \cap \{5, 6, 7, 9, 10\} \\ &= \{ \}\end{aligned}$$

$$A \cap (B \cap C) = \{1, 2, 3, 4\} \cap \{5, 6, 7\} = \{ \}$$

R.H.S.

$$\begin{aligned}A \cap (B \cap C) &= \{1, 2, 3, 4\} \cap (\{3, 4, 5, 6, 7, 8\} \cap \{5, 6, 7, 9, 10\}) \\ &= \{1, 2, 3, 4\} \cap \{5, 6, 7\} \\ &= \{ \}\end{aligned}$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$



(b) $A = \phi$, $B = \{0\}$, $C = \{0, 1, 2\}$

To show $(A \cap B) \cap C = A \cap (B \cap C)$

L.H.S.

$$\begin{aligned}(A \cap B) \cap C &= (\phi \cap \{0\}) \cap \{0, 1, 2\} \\ &= \{ \} \cap \{0, 1, 2\} \\ &= \{ \}\end{aligned}$$

R.H.S.

$$\begin{aligned}A \cap (B \cap C) &= \{ \} \cap (\{0\} \cap \{0, 1, 2\}) \\ &= \{ \} \cap \{0\} \\ &= \{ \}\end{aligned}$$

$$\Rightarrow (A \cap B) \cap C = A \cap (B \cap C)$$

(c) N, Z, Q

To show $(N \cap Z) \cap Q = N \cap (Z \cap Q)$

L.H.S.

$$(N \cap Z) \cap Q = N \cap Q = N$$

R.H.S.

$$N \cap (Z \cap Q) = N \cap Z = N$$

$$\Rightarrow (N \cap Z) \cap Q = N \cap (Z \cap Q)$$

(iii) **Distributivity of Union over intersection**

(a) $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6, 7, 8\}$, $C = \{5, 6, 7, 9, 10\}$

Distributivity of Union over intersection

i.e. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

L.H.S.

$$\begin{aligned}A \cup (B \cap C) &= \{1, 2, 3, 4\} \cup (\{3, 4, 5, 6, 7, 8\} \cap \{5, 6, 7, 9, 10\}) \\ &= \{1, 2, 3, 4\} \cup \{5, 6, 7\} \\ &= \{1, 2, 3, 4, 5, 6, 7\}\end{aligned}$$

R.H.S.

$$\begin{aligned}(A \cup B) \cap (A \cup C) &= (\{1, 2, 3, 4\} \cup \{3, 4, 5, 6, 7, 8\}) \cap (\{1, 2, 3, 4\} \cup \{5, 6, 7, 9, 10\}) \\ &= \{1, 2, 3, 4, 5, 6, 7, 8\} \cap \{1, 2, 3, 4, 5, 6, 7, 9, 10\} \\ &= \{1, 2, 3, 4, 5, 6, 7\}\end{aligned}$$

$$\Rightarrow A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$



(b) $A = \phi$, $B = \{0\}$, $C = \{0, 1, 2\}$

To show $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

L.H.S.

$$\begin{aligned} A \cup (B \cap C) &= \{ \} \cup (\{0\} \cap \{0, 1, 2\}) \\ &= \{ \} \cup \{0\} = \{0\} \end{aligned}$$

R.H.S.

$$\begin{aligned} (A \cup B) \cap (A \cup C) &= (\{ \} \cup \{0\}) \cap (\{ \} \cup \{0, 1, 2\}) \\ &= \{0\} \cap \{0, 1, 2\} = \{0\} \end{aligned}$$

$$\Rightarrow A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

(c) N, Z, Q

To show $N \cup (Z \cap Q) = (N \cup Z) \cap (N \cup Q)$

L.H.S.

$$N \cup (Z \cap Q) = N \cup Z = Z$$

R.H.S.

$$(N \cup Z) \cap (N \cup Q) = Z \cap Q = Z$$

$$\Rightarrow N \cup (Z \cap Q) = (N \cup Z) \cap (N \cup Q)$$

(iv) **Distributivity of intersection over union.**

(a) $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6, 7, 8\}$, $C = \{5, 6, 7, 9, 10\}$

Distributivity of intersection over union

$$\text{i.e. } A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

To show

L.H.S.

$$\begin{aligned} A \cap (B \cup C) &= \{1, 2, 3, 4\} \cap (\{3, 4, 5, 6, 7, 8\} \cup \{5, 6, 7, 9, 10\}) \\ &= \{1, 2, 3, 4\} \cap \{3, 4, 5, 6, 7, 8, 9, 10\} \\ &= \{3, 4\} \end{aligned}$$

R.H.S.

$$\begin{aligned} (A \cap B) \cup (A \cap C) &= (\{1, 2, 3, 4\} \cap \{3, 4, 5, 6, 7, 8\}) \cup (\{1, 2, 3, 4\} \cap \{5, 6, 7, 9, 10\}) \\ &= \{3, 4\} \cup \{ \} \\ &= \{3, 4\} \end{aligned}$$

$$\Rightarrow A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$



(b) $A = \phi$, $B = \{0\}$, $C = \{0, 1, 2\}$

To show $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

L.H.S.

$$\begin{aligned} A \cap (B \cup C) &= \{ \} \cap (\{0\} \cup \{0, 1, 2\}) \\ &= \{ \} \cap \{0, 1, 2\} = \{ \} \end{aligned}$$

R.H.S.

$$\begin{aligned} (A \cap B) \cup (A \cap C) &= (\{ \} \cap \{0\}) \cup (\{ \} \cap \{0, 1, 2\}) \\ &= \{ \} \cup \{ \} \\ &= \{ \} \end{aligned}$$

$$\Rightarrow A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

(c) N, Z, Q

To show $N \cap (Z \cup Q) = (N \cap Z) \cup (N \cap Q)$

L.H.S.

$$N \cap (Z \cup Q) = N \cap Z = N$$

R.H.S.

$$(N \cap Z) \cup (N \cap Q) = N \cup N = N$$

$$\Rightarrow N \cap (Z \cup Q) = (N \cap Z) \cup (N \cap Q)$$

Q.3 Verify De-Morgan's Laws for the following sets

$$U = \{1, 2, 3, \dots, 20\}, \quad A = \{2, 4, 6, \dots, 20\}, \quad B = \{1, 3, 5, \dots, 19\}.$$

Solution:

De Morgan's Laws are

$$(A \cup B)' = A' \cap B'$$

$$(A \cap B)' = A' \cup B'$$

$$\begin{aligned} A \cup B &= \{2, 4, 6, \dots, 20\} \cup \{1, 3, 5, \dots, 19\} \\ &= \{1, 2, 3, \dots, 20\} \end{aligned}$$

$$\begin{aligned} (A \cup B)' &= U - (A \cup B) \\ &= U - \{1, 2, 3, \dots, 20\} \\ &= \{1, 2, 3, \dots, 20\} - \{1, 2, 3, \dots, 20\} \\ &= \{ \} \quad \dots \dots \dots (1) \end{aligned}$$



$$\text{Now, } A' = U - A$$

$$= \{1, 2, 3, \dots, 20\} - \{2, 4, 6, \dots, 20\}$$

$$= \{1, 3, 5, \dots, 19\}$$

$$B' = U - B$$

$$= \{1, 2, 3, \dots, 20\} - \{1, 3, 5, \dots, 19\}$$

$$= \{2, 4, 6, \dots, 20\}$$

$$A' \cap B' = \{1, 3, 5, \dots, 19\} \cap \{2, 4, 6, \dots, 20\}$$

$$= \{ \} \quad \dots (2)$$

From equations (1) and (2) it is clear that

$$(A \cup B)' = A' \cap B'$$

$$\text{Now } A \cap B = \{2, 4, 6, \dots, 20\} \cap \{1, 3, 5, \dots, 19\}$$

$$= \{ \}$$

$$(A \cap B)' = U - (A \cap B)$$

$$= \{1, 2, 3, \dots, 20\} - \{ \}$$

$$= \{1, 2, 3, \dots, 20\} \quad \dots (3)$$

$$A' = U - A$$

$$= \{1, 2, 3, \dots, 20\} - \{2, 4, 6, \dots, 20\}$$

$$= \{1, 3, 5, \dots, 19\}$$

$$B' = U - B$$

$$= \{1, 2, 3, \dots, 20\} - \{1, 3, 5, \dots, 19\}$$

$$= \{2, 4, 6, \dots, 20\}$$

$$A' \cup B' = \{1, 3, 5, \dots, 19\} \cup \{2, 4, 6, \dots, 20\}$$

$$= \{1, 2, 3, 4, \dots, 20\} \quad \dots (4)$$

From equations (3) and (4) it is clear that $(A \cup B)' = A' \cup B'$.

Q.4 Let U = the set of all the English alphabet.

$A = \{x \mid x \text{ is a vowel}\}, \quad B = \{y \mid y \text{ is a consonant}\}$

Verify De-Morgan's Laws for these sets.

Solution:

De-Morgan's Law are

$$(A \cup B)' = A' \cap B' \text{ and}$$

$$(A \cap B)' = A' \cup B'$$



$$A \cup B = \{x \mid x \text{ is a vowel}\} \cup \{y \mid y \text{ is a consonant}\}$$

$$= \text{The set of the English alphabet.}$$

$$(A \cup B)' = U - A \cup B$$

$$= \text{The set of English alphabet} - \text{The set of English alphabet}$$

$$= \{ \}$$

..... (1)

$$A' = U - A$$

$$= \text{The set of English alphabet} - \{x \mid x \text{ is a vowel}\}$$

$$= \{y \mid y \text{ is a consonant}\}$$

$$B' = U - B$$

$$= \text{The set of English alphabet} - \{y \mid y \text{ is a consonant}\}$$

$$= \{x \mid x \text{ is a vowel}\}$$

$$A' \cap B' = \{y \mid y \text{ is a consonant}\} \cap \{x \mid x \text{ is a vowel}\}$$

$$= \{ \}$$

..... (2)

From equations (1) and (2) $(A \cup B)' = A' \cap B'$.

Now $A \cap B = \{x \mid x \text{ is a vowel}\} \cap \{y \mid y \text{ is a consonant}\}$

$$= \{ \}$$

$$(A \cap B)' = U - A \cap B$$

$$= \text{The set of English alphabet} - \{ \}$$

$$= \text{The set of English alphabet} \quad \text{..... (1)}$$

$$A' = U - A$$

$$= \text{The set of English alphabet} - \{x \mid x \text{ is a vowel}\}$$

$$= \{y \mid y \text{ is a consonant}\}$$

$$B' = U - B$$

$$= \text{The set of English alphabet} - \{y \mid y \text{ is a consonant}\}$$

$$= \{x \mid x \text{ is a vowel}\}$$

$$A' \cup B' = \{y \mid y \text{ is a consonant}\} \cup \{x \mid x \text{ is a vowel}\}$$

$$= \text{The set of English alphabet} \quad \text{..... (2)}$$

From equations (1) and (2).

It is clear that $(A \cap B)' = A' \cup B'$.



Q.5 With help of Venn Diagram, verify the two distributive laws in the following sets w.r.t. union and intersection.

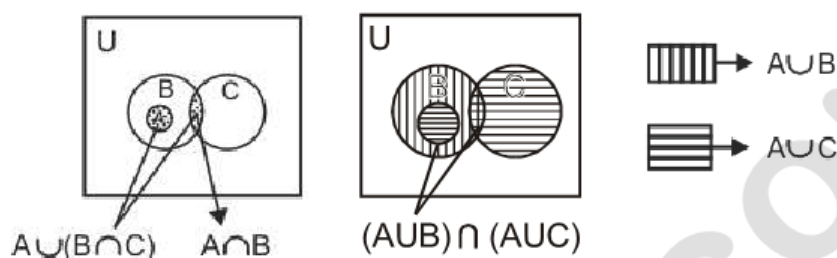
- (i) $A \subseteq B$, $A \cap C = \phi$ and B and C are overlapping.
 (ii) A and B are overlapping, B and C are overlapping but A and C are disjoint.

Solution:

- (i) $A \subseteq B$, $A \cap C = \phi$ and B and C are overlapping

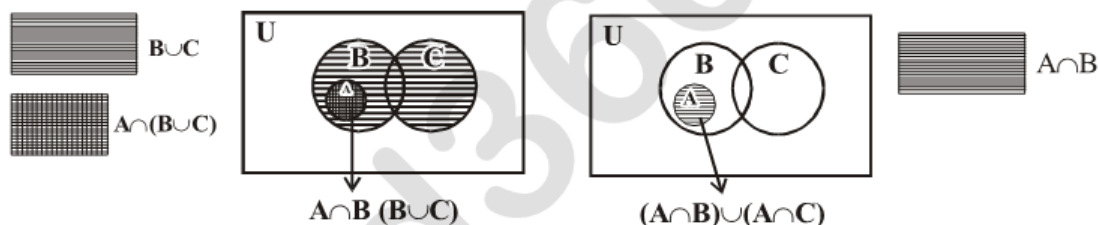
Distributivity of union over intersection

i.e. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$



Distributivity of intersection over union i.e.

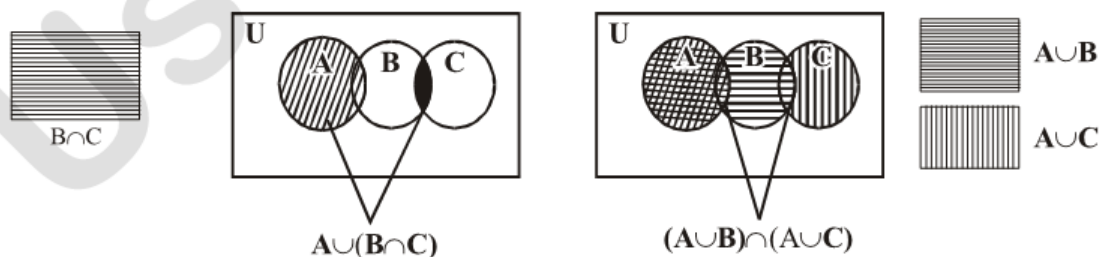
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$



- (ii) A and B are overlapping, B and C are overlapping, but A and C are disjoint.

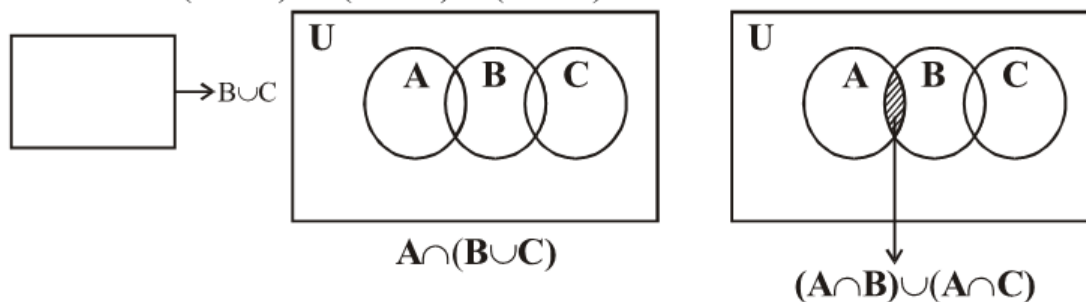
Distributivity of union over intersection

i.e. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$



Distributivity of intersection over union.

i.e. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$



Q.6 Taking any set, say $A = \{1, 2, 3, 4, 5\}$ verify the following

(i) $A \cup \phi = A$ (ii) $A \cup A = A$ (iii) $A \cap A = A$

Solution:

(i) $A \cup \phi = A$

$$A = \{1, 2, 3, 4, 5\}$$

$$\begin{aligned} A \cup \phi &= \{1, 2, 3, 4, 5\} \cup \{ \} \\ &= \{1, 2, 3, 4, 5\} \\ &= A \end{aligned}$$

$$\Rightarrow A \cup \phi = A$$

(ii) $A \cup A = A$

$$A = \{1, 2, 3, 4, 5\}$$

$$\begin{aligned} A \cup A &= \{1, 2, 3, 4, 5\} \cup \{1, 2, 3, 4, 5\} \\ &= \{1, 2, 3, 4, 5\} = A \end{aligned}$$

$$\Rightarrow A \cup A = A$$

(iii) $A \cap A = A$

$$A = \{1, 2, 3, 4, 5\}$$

$$\begin{aligned} A \cap A &= \{1, 2, 3, 4, 5\} \cap \{1, 2, 3, 4, 5\} \\ &= \{1, 2, 3, 4, 5\} = A \end{aligned}$$

$$\Rightarrow A \cap A = A$$

Q.7 If $U = \{1, 2, 3, 4, 5, \dots, 20\}$, $A = \{1, 3, 5, \dots, 19\}$ verify the following

(i) $A \cup A' = U$ (ii) $A \cap U = A$ (iii) $A \cap A' = \phi$

Solution:

(i) $A \cup A' = U$

$$\begin{aligned} A' &= U - A = \{1, 2, 3, 4, 5, \dots, 20\} - \{1, 3, 5, \dots, 19\} \\ &= \{2, 4, 6, 8, \dots, 20\} \end{aligned}$$

$$\begin{aligned} \Rightarrow A \cup A' &= \{1, 3, 5, \dots, 19\} \cup \{2, 4, 6, \dots, 20\} \\ &= \{1, 2, 3, 4, 5, \dots, 20\} = U \end{aligned}$$

$$\Rightarrow A \cup A' = U$$



$$(ii) \quad A \cap U = A$$

$$\begin{aligned} A \cap U &= \{1, 3, 5, \dots, 19\} \cap \{1, 2, 3, 4, 5, \dots, 20\} \\ &= \{1, 3, 5, \dots, 19\} = A \end{aligned}$$

$$\Rightarrow A \cap U = A$$

$$(iii) \quad A \cap A' = \phi$$

$$\begin{aligned} A' &= U - A \\ &= \{1, 2, 3, 4, 5, \dots, 20\} - \{1, 3, 5, \dots, 19\} \\ &= \{2, 4, 6, \dots, 20\} \end{aligned}$$

$$\begin{aligned} A \cap A' &= \{1, 3, 5, \dots, 19\} \cap \{2, 4, 6, \dots, 20\} \\ &= \{ \} = \phi \end{aligned}$$

$$\Rightarrow A \cap A' = \phi$$

Q.8 From suitable properties of union and intersection deduce the following results:

$$(i) \quad A \cap (A \cup B) = A \cup (A \cap B)$$

(Lahore Board 2007, 2010)

$$(ii) \quad A \cup (A \cap B) = A \cap (A \cup B)$$

(Gujranwala Board 2003)

Solution:

$$(i) \quad A \cap (A \cup B) = A \cup (A \cap B)$$

$$\begin{aligned} \text{L.H.S.} &= A \cap (A \cup B) \\ &= (A \cap A) \cup (A \cap B) && \text{Using distributive law} \\ &= A \cup (A \cap B) && \text{Since } A \cap A = A \\ &= \text{R.H.S.} \end{aligned}$$

$$(ii) \quad A \cup (A \cap B) = A \cap (A \cup B)$$

$$\begin{aligned} \text{L.H.S.} &= A \cup (A \cap B) \\ &= (A \cup A) \cap (A \cup B) && \text{by distributive law} \\ &= A \cap (A \cup B) && \text{Since } A \cup A = A \\ &= \text{R.H.S.} \end{aligned}$$

Q.9 Using Venn Diagram, verify the following results.

$$(i) \quad A \cap B' = A \text{ iff } A \cap B = \phi$$

$$(ii) \quad (A - B) \cup B = A \cup B$$

$$(iii) \quad (A - B) \cap B = \phi$$

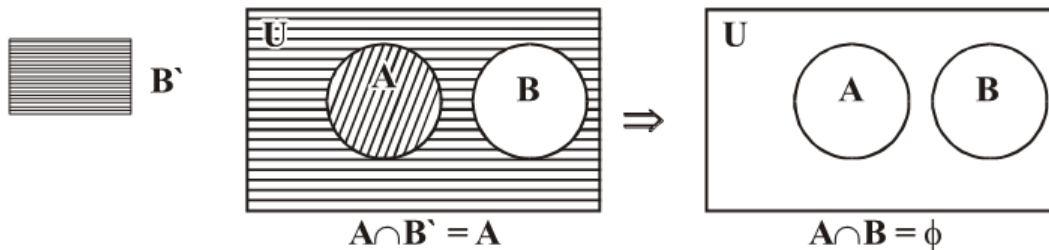
$$(iv) \quad A \cup B = A \cup (A' \cap B')$$



Solution:

(i) $A \cap B' = A$ iff $A \cap B = \phi$

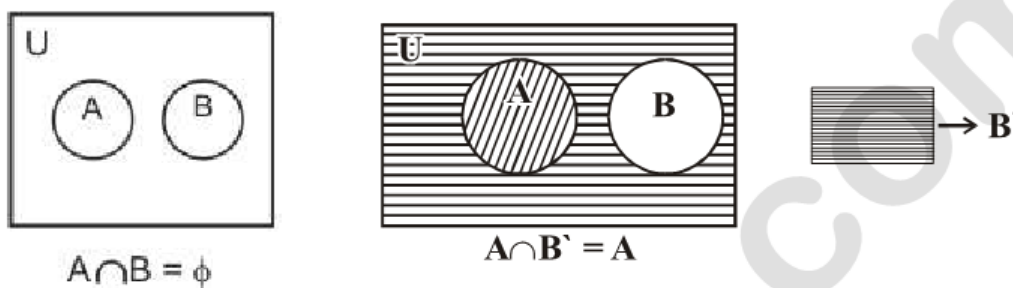
i.e. by Venn diagram



on contrary;

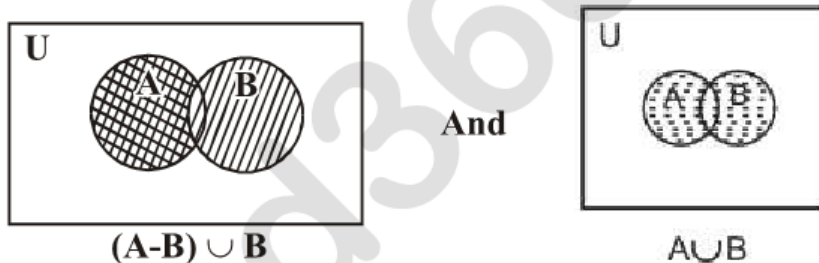
Suppose $A \cap B = \phi$

i.e.



$\Rightarrow A \cap B' = A$ if $A \cap B = \phi$

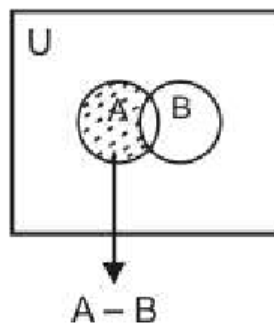
(ii) $(A - B) \cup B = A \cup B$



From above figures it is clear that

$(A - B) \cup B = A \cup B$

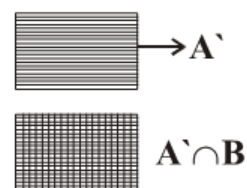
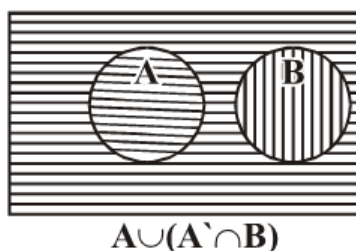
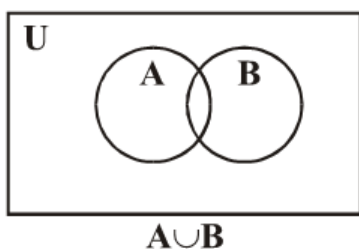
(iii) $(A - B) \cap B = \phi$



It is clear from above figure that $(A - B) \cap B = \phi$



(iv) $A \cup B = A \cup (A' \cap B)$



In above two figures the shaded portion is same.

$\Rightarrow A \cup B = A \cup (A' \cap B)$

INDUCTIVE AND DEDUCTIVE LOGIC

Induction

The way of drawing conclusions on the basis of a few basic experiments or observations is called induction.

Deduction

The way of drawing conclusions by accepting some well known facts is called deduction.

Proposition

A declarative statement which may be true or false but not both is called proposition.

Aristotelian and non-Aristotelian Logics:

Deductive logic in which every statement is regarded as true or false and there is no other possibility, is called Aristotelian logic.

Logic in which there is scope for a third or fourth possibility is called non-Aristotelian logic.

Symbolic Logic

Symbol	How to be read	Symbolic expression	How to be read
$\sim$ (Negation)	not	$\equiv p$	Not p
$\wedge$ (Conjunction)	and	$p \wedge q$	p and q
$\vee$ (Disjunction)	or	$p \vee q$	p or q
$\rightarrow$ (Conditional)	if then implies	$p \rightarrow q$	p implies q
$\leftrightarrow$ (Biconditional)	if and only if	$p \leftrightarrow q$	p if and only if q or p is equivalent to q



How to use these Logics

p	q	$\Rightarrow p$	$\Rightarrow q$	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$
T	T	F	F	T	T	T	T	T
T	F	F	T	F	T	F	T	F
F	T	T	F	F	T	T	F	F
F	F	T	T	F	F	T	T	T

Converse of $p \rightarrow q$ is $q \rightarrow p$

Inverse of $p \rightarrow q$ is $\Rightarrow p \rightarrow \Rightarrow q$

Contrapositive of $p \rightarrow q$ is $\Rightarrow q \rightarrow \Rightarrow p$

Tautology

A statement which is true for all the possible values of the variable involved in it is called a tautology.

Absurdity

A statement which is false for all the possible values of the variable involved in it is called an Absurdity.

Contingency

A statement which can be true or false depending upon the truth values of the variable involved in it is called contingency.



EXERCISE 2.4

Q.1 Write the converse, inverse and contrapositive of the following conditionals.

(i) $\sim p \rightarrow q$

(Gujranwala Board 2007)

(ii) $q \rightarrow p$

(iii) $\sim p \rightarrow \sim q$

(Gujranwala Board 2003, 2007)

(iv) $\sim q \rightarrow \sim p$

Solution:

	Given Conditional	Converse	Inverse	Contrapositive
(i)	$\sim p \rightarrow q$	$q \rightarrow \sim p$	$p \rightarrow \sim q$	$\sim q \rightarrow p$
(ii)	$q \rightarrow p$	$p \rightarrow q$	$\sim q \rightarrow \sim p$	$\sim p \rightarrow \sim q$
(iii)	$\sim p \rightarrow \sim q$	$\sim q \rightarrow \sim p$	$p \rightarrow q$	$q \rightarrow p$
(iv)	$\sim q \rightarrow \sim p$	$\sim p \rightarrow \sim q$	$q \rightarrow p$	$P \rightarrow q$



(i)

				Given Conditional	Converse	Inverse	Contrapositive
p	q	$\sim p$	$\sim q$	$\sim p \rightarrow q$	$q \rightarrow \sim p$	$p \rightarrow \sim q$	$\sim q \rightarrow p$
T	T	F	F	T	F	F	T
T	F	F	T	T	T	T	T
F	T	T	F	T	T	F	T
F	F	T	T	F	T	T	F

(ii)

				Given Conditional	Converse	Inverse	Contrapositive
p	q	$\sim p$	$\sim q$	$q \rightarrow p$	$p \rightarrow q$	$\sim q \rightarrow \sim p$	$\sim p \rightarrow \sim q$
T	T	F	F	T	T	T	T
T	F	F	T	T	F	F	T
F	T	T	F	F	T	T	F
F	F	T	T	T	T	T	T

(iii)

				Given Conditional	Converse	Inverse	Contrapositive
p	q	$\sim p$	$\sim q$	$\sim p \rightarrow \sim q$	$\sim q \rightarrow \sim p$	$p \rightarrow q$	$q \rightarrow p$
T	T	F	F	T	T	T	T
T	F	F	T	T	F	F	T
F	T	T	F	F	T	T	F
F	F	T	T	T	T	T	T

(iv)

				Given Conditional	Converse	Inverse	Contrapositive
p	q	$\sim p$	$\sim q$	$\sim q \rightarrow \sim p$	$\sim p \rightarrow \sim q$	$q \rightarrow p$	$p \rightarrow q$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T



Q.2 Construct truth tables for the following statements

(i) $(p \rightarrow \sim p) \vee (p \rightarrow q)$ (ii) $(p \wedge \sim p) \rightarrow q$ (Gujranwala Board 2003)

(iii) $\sim(p \rightarrow q) \leftrightarrow (p \wedge \sim q)$

Solution:

(i) $(p \rightarrow \sim p) \vee (p \rightarrow q)$

Required truth table is given below

P	q	$\sim p$	$p \rightarrow \sim p$	$p \rightarrow q$	$(p \rightarrow \sim p) \vee (p \rightarrow q)$
T	T	F	F	T	T
T	F	F	F	F	F
F	T	T	T	T	T
F	F	T	T	T	T

(ii) $(p \wedge \sim p) \rightarrow q$

Required truth table is given below

p	q	$\sim p$	$p \wedge \sim p$	$(p \wedge \sim p) \rightarrow q$
T	T	F	F	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

(iii) $\sim(p \rightarrow q) \leftrightarrow (p \wedge \sim q)$

Required truth table is given below

P	q	$p \rightarrow q$	$\sim(p \rightarrow q)$	$\sim q$	$p \wedge \sim q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

As entries in the columns of $\sim(p \rightarrow q)$ and $p \wedge \sim q$ are same.

$$\Rightarrow \sim(p \rightarrow q) \leftrightarrow p \wedge \sim q$$



Q.3 Show that each of the following statement is a tautology:

(i) $(p \wedge q) \rightarrow p$ (ii) $p \rightarrow (p \vee q)$

(Lahore Board 2004, 2009)

(iii) $\sim(p \rightarrow q) \rightarrow p$ (iv) $\sim q \wedge (p \rightarrow q) \rightarrow \sim p$

Solution:

(i) $(p \wedge q) \rightarrow p$

Truth table of $(p \wedge q) \rightarrow p$ is given below

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

As $(p \wedge q) \rightarrow p$ is true for all values of the variable so it is a tautology.

(ii) $p \rightarrow (p \vee q)$

Truth table of $p \rightarrow (p \vee q)$ is given below

p	q	$p \vee q$	$p \rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

Since $P \rightarrow (P \vee q)$ is true for all values of the variable involved, so it is a tautology.

(iii) $(p \rightarrow q) \rightarrow P$

Truth table of $\sim(p \rightarrow q) \rightarrow P$ is given below.

p	q	$p \rightarrow q$	$\sim(p \rightarrow q)$	$\sim(p \rightarrow q) \rightarrow p$
T	T	T	F	T
T	F	F	T	T
F	T	T	F	T
F	F	T	F	T

As $\Rightarrow (p \rightarrow q) \rightarrow p$ is true for all values of the variable so it is a tautology.



(iv) $\sim q \wedge (p \rightarrow q) \rightarrow \sim p$

Truth table of $\sim q \wedge (p \rightarrow q) \rightarrow \sim p$ is

p	q	$\sim p$	$\sim q$	$(p \rightarrow q)$	$\sim q \wedge (p \rightarrow q)$	$\sim q \wedge (p \rightarrow q) \rightarrow \sim p$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

As $\sim q \wedge (p \rightarrow q) \rightarrow \sim p$ is true for all values of the variable so it is a tautology.**Q.4 Determine whether each of the following is a tautology, a contingency, or an absurdity.**

(i) $p \wedge \sim p$ (ii) $p \rightarrow (q \rightarrow p)$ (iii) $q \vee (\sim q \vee p)$

(Gujranwala Board 2005)

Solution:

(i) $p \wedge \sim p$

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

As $p \wedge \sim p$ is false for all values. So it is an absurdity.

(ii) $p \rightarrow (q \rightarrow p)$

Truth table of $p \rightarrow (q \rightarrow p)$ is

p	q	$q \rightarrow p$	$p \rightarrow (q \rightarrow p)$
T	T	T	T
T	F	T	T
F	T	F	T
F	F	T	T

As $p \rightarrow (q \rightarrow p)$ is true for all values of the variable so it is a tautology.

(iii) $q \vee (\sim q \vee p)$

Truth table of $q \vee (\sim q \vee p)$ is

p	q	$\sim q$	$(\sim q \vee p)$	$q \vee (\sim q \vee p)$
T	T	F	T	T
T	F	T	T	T
F	T	F	F	T
F	F	T	T	T

As $q \vee (\sim q \vee p)$ is true for all the values of the variable so it is a tautology.

Q.5 Prove that $p \vee (\sim p \wedge \sim q) \vee (p \wedge q) = p \vee (\sim p \wedge \sim q)$ (Lahore Board 2005)

Solution:

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$p \vee (\sim p \wedge \sim q)$	$p \wedge q$	$p \vee (\sim p \wedge \sim q) \vee (p \wedge q)$
T	T	F	F	F	T	T	F
T	F	F	F	F	T	F	T
F	T	T	T	F	F	F	T
F	F	T	T	T	T	F	T

As entries in the columns of $p \vee (\sim p \wedge \sim q) \vee (p \wedge q)$ and $p \vee (\sim p \wedge \sim q)$ are same.

$$\Rightarrow p \vee (\sim p \wedge \sim q) \vee (p \wedge q) = p \vee (\sim p \wedge \sim q)$$

Hence proved.



EXERCISE 2.5

Convert the following theorems to logical and prove them by constructing truth tables.

Q.1 $(A \cap B)' = A' \cup B'$

(Lahore Board 2004)

Solution:

Its logical form is $\sim (p \wedge q) = \sim p \vee \sim q$ its truth table is given below

p	q	$p \wedge q$	$\sim (p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

As entries in the columns of $\sim (p \wedge q)$ and $\sim p \vee \sim q$ are same.

$$\Rightarrow \sim (p \wedge q) = \sim p \vee \sim q$$

$$\Rightarrow (A \cap B)' = A' \cup B'$$



Q.2 $(A \cup B)' = A' \cap B'$ **Solution:**

$$(A \cup B) \cup C = A \cup (B \cup C)$$

Its logical form is $(p \vee q) \vee r = p \vee (q \vee r)$

Its truth table is given below

p	q	r	$p \vee q$	$(p \vee q) \vee r$	$q \vee r$	$p \vee (q \vee r)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
F	T	T	T	T	T	T
F	F	T	F	T	T	T
F	T	F	T	T	T	T
T	F	F	T	T	F	T
F	F	F	F	F	F	F

As the entries in columns of $(p \vee q) \vee r$ and $p \vee (q \vee r)$ are same.

$$\Rightarrow (p \vee q) \vee r = p \vee (q \vee r)$$

$$\Rightarrow (A \cup B) \cup C = A \cup (B \cup C)$$

Hence proved.

Q.3 $(A \cap B) \cap C = A \cap (B \cap C)$ **Solution:**

$$(A \cap B) \cap C = A \cap (B \cap C)$$

Its logical form is $(p \wedge q) \wedge r = p \wedge (q \wedge r)$.

Its truth table is given below.

p	q	r	$p \wedge q$	$(p \wedge q) \wedge r$	$(q \wedge r)$	$p \wedge (q \wedge r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
F	T	T	F	F	T	F
F	F	T	F	F	F	F
F	T	F	F	F	F	F
T	F	F	F	F	F	F
F	F	F	F	F	F	F

As entries in the columns of $(p \wedge q) \wedge r$ and $p \wedge (q \wedge r)$ are same.

$$\Rightarrow (p \wedge q) \wedge r = p \wedge (q \wedge r)$$

$$\Rightarrow (A \cap B) \cap C = A \cap (B \cap C)$$



Q.4 $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ **Solution:**

Its logical form is $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$. Its truth table is given below.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
F	T	T	T	T	T	T	T
F	F	T	F	F	F	T	F
F	T	F	F	F	T	F	F
T	F	F	F	T	T	T	T
F	F	F	F	F	F	F	F

As entries in the columns of $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are same. So

$$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$$

or

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Hence proved.



EXERCISE 2.6

Binary Relation

Let A and B be two non-empty sets, then any subset of Cartesian product $A \times B$ is called a binary relation, or simply a relation from A to B .

Q.1 For $A = \{1, 2, 3, 4\}$, find the following relation in A . State the domain and range of each relation. Also draw the graph of each.

(i) $\{(x, y) \mid y = x\}$ (Lahore Board 2010)

(ii) $\{(x, y) \mid y + x = 5\}$

(iii) $\{(x, y) \mid x + y < 5\}$ (Lahore Board 2011)

(iv) $\{(x, y) \mid x + y > 5\}$ (Gujranwala Board 2003, Lahore Board 2003)

Solution:

Given that

$$A = \{1, 2, 3, 4\}$$

Then

$$\begin{aligned} A \times A &= \{1, 2, 3, 4\} \times \{1, 2, 3, 4\} \\ &= \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), \\ &\quad (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)\} \end{aligned}$$



(i) $\{(x, y) \mid y = x\}$

$$\begin{aligned}\text{Let } r_1 &= \{(x, y) \mid y = x\} \\ &= \{(1, 1), (2, 2), (3, 3), (4, 4)\}\end{aligned}$$

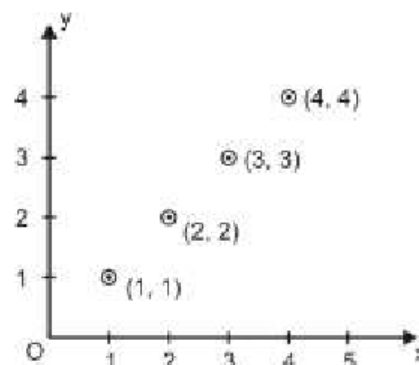
$\therefore r_1$ is the required relation.

Now

$$\text{Domain } r_1 = \{1, 2, 3, 4\}$$

$$\text{Range } r_1 = \{1, 2, 3, 4\}$$

and graph of r_1 is given



(ii) $\{(x, y) \mid y + x = 5\}$

$$\begin{aligned}\text{Let } r_2 &= \{(x, y) \mid y + x = 5\} \\ &= \{(1, 4), (2, 3), (3, 2), (4, 1)\}\end{aligned}$$

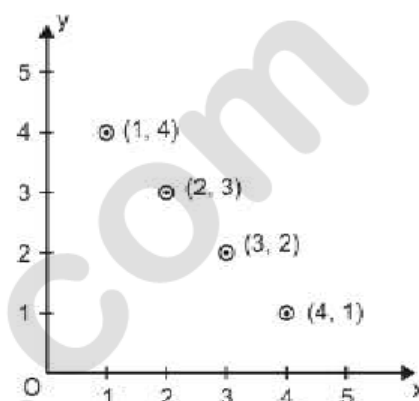
$\Rightarrow r_2$ is the required relation.

Now

$$\text{Domain } r_2 = \{1, 2, 3, 4\}$$

$$\text{Range } r_2 = \{4, 3, 2, 1\}$$

and its graph is given



(iii) $\{(x, y) \mid x + y < 5\}$

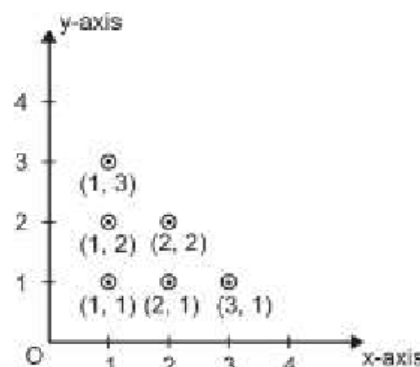
$$\begin{aligned}\text{Let } r_3 &= \{(x, y) \mid x + y < 5\} \\ &= \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1)\}\end{aligned}$$

$\therefore r_3$ is the required relation.

$$\text{Domain } r_3 = \{1, 2, 3\}$$

$$\text{Range } r_3 = \{1, 2, 3\}$$

Its graph is given



(iv) $\{(x, y) \mid x + y > 5\}$

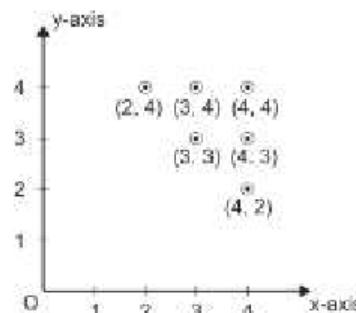
$$\begin{aligned}\text{Let } r_4 &= \{(x, y) \mid x + y > 5\} \\ &= \{(2, 4), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)\}\end{aligned}$$

$\therefore r_4$ is the required relation.

$$\text{Domain } r_4 = \{2, 3, 4\}$$

$$\text{Range } r_4 = \{4, 3, 2\}$$

Its graph is given



Q.2 Repeat Q.1 when $A = \mathbb{R}$, the set of real numbers. Which of the real lines are functions.

(i) $A = \mathbb{R} \quad \{(x, y) \mid y = x\}$

(ii) $\{(x, y) \mid y + x = 5\}$

(iii) $\{(x, y) \mid x + y < 5\}$

(iv) $\{(x, y) \mid x + y > 5\}$

Solution:

(i) $\{(x, y) \mid y = x\}$

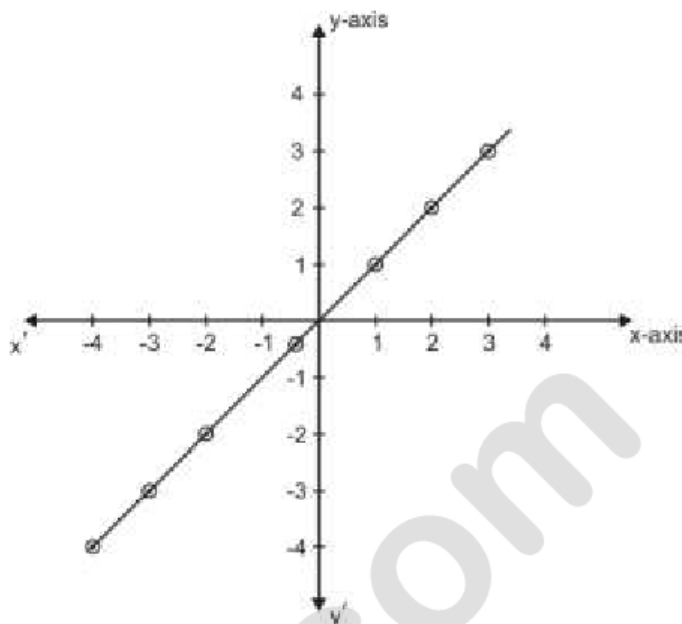
Let r_1 be the required relation, then

$$r_1 = \{(x, y) \mid y = x\}$$

$$\text{Dom } r_1 = \mathbb{R}$$

$$\text{Range } r_1 = \mathbb{R}$$

Its graph is given



Its graph will be a straight line and r_1 is a function, because any vertical line will cut it at only one point.

(ii) $\{(x, y) \mid y + x = 5\}$

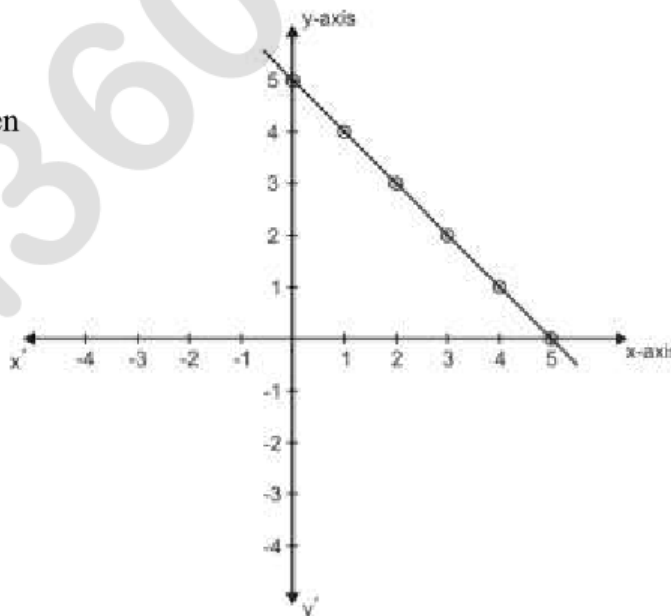
Let r_2 be the required relation, then

$$r_2 = \{(x, y) \mid y + x = 5\}$$

$$\text{Dom } r_2 = \mathbb{R}$$

$$\text{Range } r_2 = \mathbb{R}$$

Its graph will be



Its graph will be a straight line and it is a function because any vertical line will cut it at only one point.



(iii) $\{(x, y) \mid x + y < 5\}$

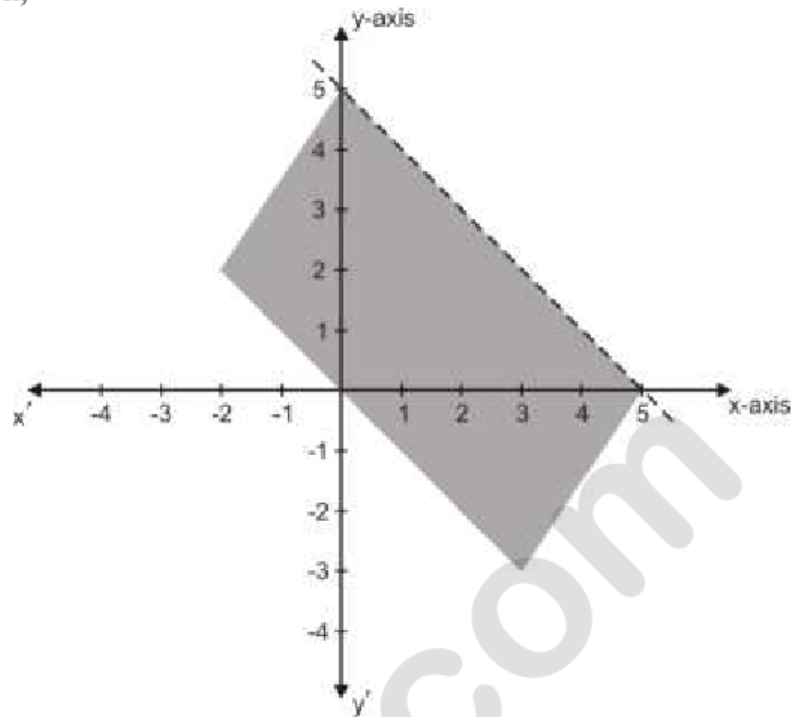
Let r_3 be the required relation,

then $r_3 = \{(x, y) \mid x + y < 5\}$

Dom $r_3 = \mathbb{R}$

Range $r_3 = \mathbb{R}$

Its graph is given below



It is not a function because any vertical line does not cut it at only one point.

(iv) $\{(x, y) \mid x + y > 5\}$

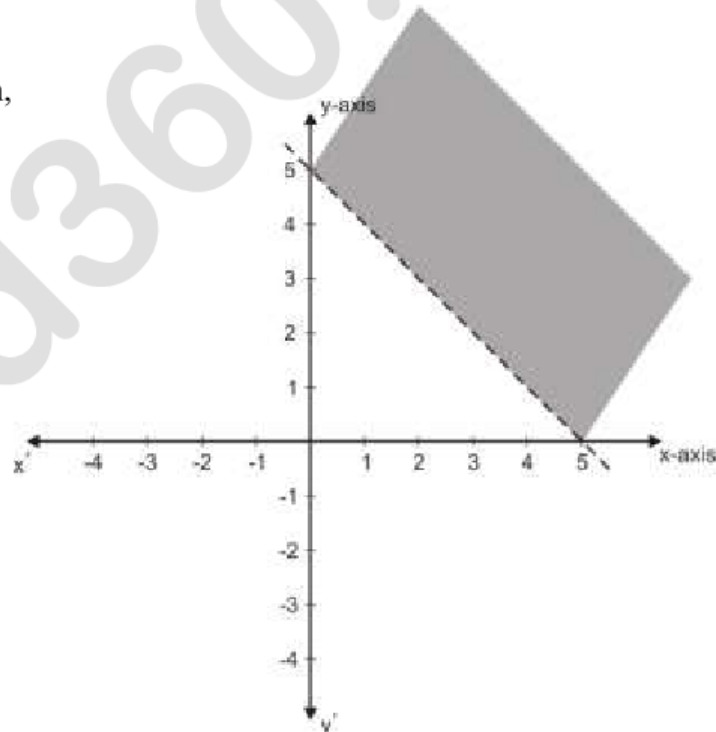
Let r_4 be the required relation,

then $r_4 = \{(x, y) \mid x + y > 5\}$

Dom $r_4 = \mathbb{R}$ and

Range $r_4 = \mathbb{R}$

Its graph is given



It is not a function because any vertical line does not cut it at only one point.



Function

A binary relation 'f' is called a function if there is no repetition in the domain of f.

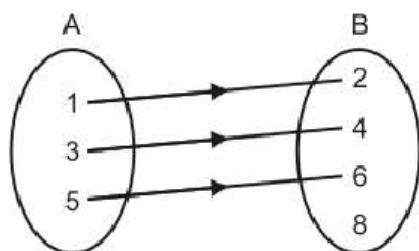
It is written as $f : A \rightarrow B$

Into Function

If a function $f : A \rightarrow B$ is such that $\text{Range } f \neq B$ then 'f' is called an Into function.

For example,

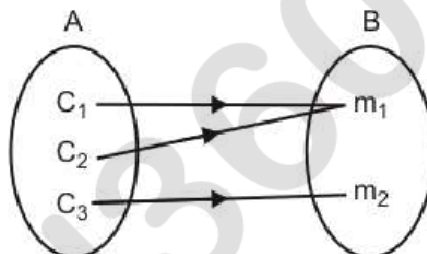
$$f = \{(1, 2), (3, 4), (5, 6)\}$$



Onto (Surjective) Function

If a function $f : A \rightarrow B$ is such that $\text{Range } f = B$ then 'f' is called an Onto (surjective) function. For example

$$f : \{(c_1, m_1), (c_2, m_1), (c_3, m_2)\}$$

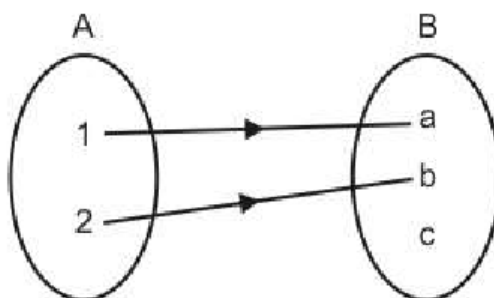


(1 – 1) and Into (Injective) Function

If f is an Into function and there is no repetition in the range of 'f' then 'f' is called an Injective (1 – 1 and into) function.

For example,

$$f : \{(1, a), (2, b)\}$$

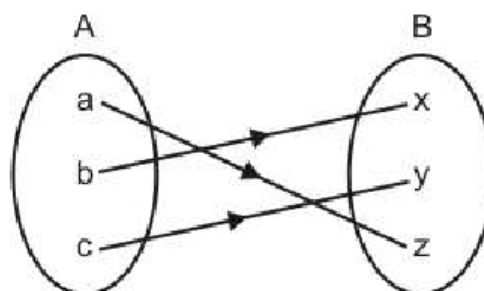


(1 – 1) and Onto (Bijective) Function

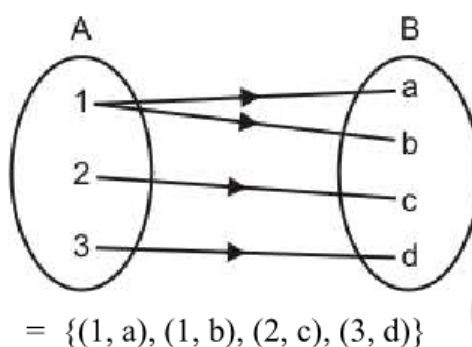
If 'f' is an onto function and there is no repetition in the range of 'f' then 'f' is called a Bijective (1 – 1 and onto) function.

For example,

$$f : \{(a, z), (b, x), (c, y)\}$$

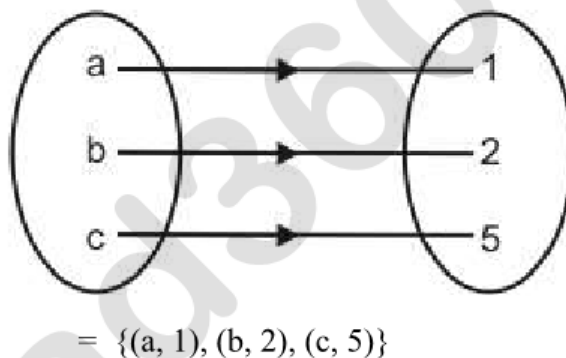


Q.3 Which of the following diagram represents a function and of which type?



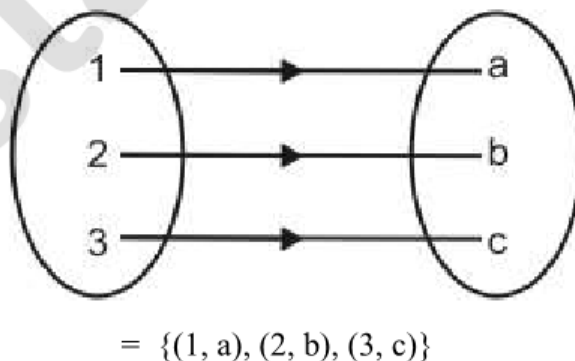
It is not a function because there is repetition in its domain.

fig (ii)



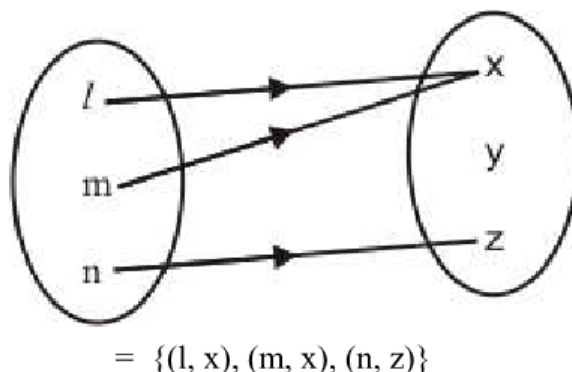
It is a function and it is 1 – 1 and onto (Bijective) function.

fig. (iii)



It is also a function and it is 1 – 1 and Onto (Bijective) function.

fig. (iv)



It is a function and it is an into function.

Inverse of A Function

Inverse of a function or relation can be obtained by interchanging the components of each ordered pair.

Q.4 Find the inverse of each of the following relations. Tell whether each relation and its inverse is a function or not.

(i) $\{(2, 1), (3, 2), (4, 3), (5, 4), (6, 5)\}$

(ii) $\{(1, 3), (2, 5), (3, 7), (4, 9), (5, 11)\}$

(iii) $\{(x, y) \mid y = 2x + 3, x \in \mathbb{R}\}$

(iv) $\{(x, y) \mid y^2 = 4ax, x \geq 0\}$

(v) $\{(x, y) \mid x^2 + y^2 = 9, |x| \leq 3, |y| \leq 3\}$

(Gujranwala Board 2003)

Solution:

(i) $\{(2, 1), (3, 2), (4, 3), (5, 4), (6, 5)\}$

Let

$$r = \{(2, 1), (3, 2), (4, 3), (5, 4), (6, 5)\}$$

There is no repetition in the domain of r .

$\Rightarrow r$ is a function

now its inverse is

$$r^{-1} = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$$

As there is no repetition in the domain of r^{-1} .

$\Rightarrow r^{-1}$ is also a function.

(ii) $\{(1, 3), (2, 5), (3, 7), (4, 9), (5, 11)\}$

Let

$$r = \{(1, 3), (2, 5), (3, 7), (4, 9), (5, 11)\}$$



As there is no repetition in the domain of r .

$\Rightarrow r$ is a function

now

$$r^{-1} = \{(3, 1), (5, 2), (7, 3), (9, 4), (11, 5)\}$$

There is no repetition in the domain of r^{-1} .

$\Rightarrow r^{-1}$ is also a function.

(iii) $\{(x, y) \mid y = 2x + 3, x \in \mathbb{R}\}$

Let $r = \{(x, y) \mid y = 2x + 3, x \in \mathbb{R}\}$

' r ' is a function because for each value of $x \in \mathbb{R}$, there is only one value of y .

Now

$$\begin{aligned} r^{-1} &= \{(x, y) \mid x = 2y + 3, x \in \mathbb{R}\} \\ &= \left\{ (x, y) \mid y = \frac{x-3}{2}, x \in \mathbb{R} \right\} \end{aligned}$$

r^{-1} is also a function because for each value of $x \in \mathbb{R}$, there is only one value of $y \in \mathbb{R}$.

(iv) $\{(x, y) \mid y^2 = 4ax, x \geq 0\}$

$$\begin{aligned} \text{Let } r &= \{(x, y) \mid y^2 = 4ax, x \geq 0\} \\ &= \{(x, y) \mid y = \pm \sqrt{4ax}, x \geq 0\} \\ &= \{(x, y) \mid y = \pm 2\sqrt{ax}, x \geq 0\} \end{aligned}$$

' r ' is not a function because for each value of x , there are two values of y .

Now

$$\begin{aligned} r^{-1} &= \{(x, y) \mid x^2 = 4ay, x \geq 0\} \\ &= \left\{ (x, y) \mid y = \frac{x^2}{4a}, x \geq 0 \right\} \end{aligned}$$

r^{-1} is a function because for each value of x , there is only one value of y .

(v) $\{(x, y) \mid x^2 + y^2 = 9, |x| \leq 3, |y| \leq 3\}$

$$\begin{aligned} \text{Let } r &= \{(x, y) \mid x^2 + y^2 = 9, |x| \leq 3, |y| \leq 3\} \\ &= \{(x, y) \mid y = \pm \sqrt{9 - x^2}, |x| \leq 3, |y| \leq 3\} \end{aligned}$$

' r ' is not a function because for each value of x there are two values of y .

Now

$$\begin{aligned} r^{-1} &= \{(x, y) \mid x^2 + y^2 = 9, |x| \leq 3, |y| \leq 3\} \\ &= \{(x, y) \mid y = \pm \sqrt{9 - x^2}, |x| \leq 3, |y| \leq 3\} \end{aligned}$$

r^{-1} is not a function because for each value of x , there are two values of y .



Binary Operation

A binary operation denoted as $*$ (read as star) on a non-empty G is a function associates with each ordered pair (a, b) , of elements of G , a unique element, denoted as $a * b$ of G .

Properties of Binary Operation

Let ' S ' be a non-empty set and $*$ a binary operation on it. Then $*$ may possess one or more of the following properties.

- (1) Commutativity: $*$ is said to be commutative if
$$a * b = b * a \quad \forall a, b \in S.$$
- (2) Associativity: $*$ is said to be associative on S
$$a * (b * c) = (a * b) * c \quad \forall a, b, c \in S$$
- (3) Existence of an identity element: An element $e \in S$ is called an identity element w.r.t. $*$ if
$$a * e = e * a = a \quad \forall a \in S.$$
- (4) Existence of inverse of each element: For any element $a \in S$, $\exists$ an element $a' \in S$ such that
$$a * a' = a' * a = e \text{ (the identity element)}$$



EXERCISE 2.7

Q.1 Complete the table, indicating by tick mark those properties which are satisfied by the specified set of number.

Property	Set of numbers	Natural	Whole	Integers	Rationale	Reals
Closure	$\oplus$	✓	✓	✓	✓	✓
	$\otimes$	✓	✓	✓	✓	✓
Associative	$\oplus$	✓	✓	✓	✓	✓
	$\otimes$	✓	✓	✓	✓	✓
Identity	$\oplus$	×	✓	✓	✓	✓
	$\otimes$	✓	✓	✓	✓	✓
Inverse	$\oplus$	×	×	✓	✓	✓
	$\otimes$	×	×	×	✓	×
Commutative	$\oplus$	✓	✓	✓	✓	✓
	$\otimes$	✓	✓	✓	✓	✓



Q.2 What are the field axioms? In what respect does the field of real numbers differ from that of complex numbers.

Solution:

A non empty set F is said to be field if

- (i) F is an abelian group w.r.t. '+'.
 - (ii) $F - \{0\}$ is an abelian group w.r.t. '×'.
 - (iii) Left and Right Distributive laws hold in F .

Field of real numbers is different from the field of complex numbers because in the field of real numbers we can compare two real numbers such that one number is less than the other, but in the field of complex numbers, we can not compare two complex numbers such that one number is less than the other.

Q.3 Show that the adjoining table is that of multiplication of the elements of the set of residue classes modulo 5.

*	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Solution:

The above table shows the multiplication of the elements of the set of residue classes modulo 5 because in table

$$2 \times 3 = 6 = 1 \text{ (remainder after dividing 6 by 5)}$$

$$2 \times 4 = 8 = 3 \text{ (remainder after dividing 8 by 5)}$$

$$3 \times 3 = 9 = 4 \text{ (remainder after dividing 9 by 5)}$$

$$3 \times 4 = 12 = 2 \text{ (remainder after dividing 12 by 5)}$$

$$4 \times 4 = 16 = 1 \text{ (remainder after dividing 16 by 5)}$$

Q.4 Prepare a table of addition of the elements of the set of residue classes modulo 4.

Solution:

Required table is given below.

Consider the set $\{0, 1, 2, 3\}$

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2



Q.5 Which of the following binary operations shown in table (a) or (b) is commutative?

*	a	b	c	d
a	a	c	b	d
b	b	c	b	a
c	c	d	b	c
d	a	a	b	b

(a)

* in table (a) is not commutative because

$$a * b = c \text{ and } b * a = b$$

⇒

$$a * b \neq b * a$$

*	a	b	c	d
a	a	c	b	d
b	c	d	b	a
c	b	b	a	c
d	d	a	c	d

(b)

* in table (b) is commutative

because

$$a * b = c \text{ and } b * a = c$$

⇒

$$a * b = b * a$$

Similarly other elements can be checked.

Q.6 Supply the missing elements of the third row of the given table. So that the operation may be associative.

*	a	b	c	d
a	a	b	c	d
b	b	a	c	d
c	—	—	—	—
d	d	c	c	d

Solution:

Suppose the required elements are p, q, r, s then

$$p = c * a = (d * b) * a = d * (b * a) = d * b = c$$

$$q = c * b = (d * b) * b = d * (b * b) = d * a = d$$

$$r = c * c = (d * b) * c = d * (b * c) = d * c = c$$

$$s = c * d = (d * b) * d = d * (b * d) = d * d = d$$



Q.7 What operation is represented by the adjoining table? Name the identity element of relevant set, if it exists. Is the operation associative? Find inverse of 0, 1, 2, 3 if they exist.

Solution:

*	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

- (i) Operation is '+' modulo 4.
- (ii) '0' is the identity element.
- (iii) yes, operation is associative.
- (iv) Inverse of 0 = 0
Inverse of 1 = 3
Inverse of 2 = 2
Inverse of 3 = 1

GROUPS

Groupoid

A closed set with respect to an operation $*$ is called a groupoid.

Semi Group

A non-empty set S is semi group if

- (i) It is closed with respect to an operation $*$
- (ii) the operation $*$ is associative.

Monoid

A semi group having an identity in it is called monoid.

Group

A monoid having inverse of each of its elements under $*$ is called a group.

Abelian Group

If a group G satisfies an additional condition i.e.

for every $a, b \in G$, $a * b = b * a$

then G is called an Abelian or commutative group under $*$.



EXERCISE 2.8

Q.1 Operation $\oplus$ performed on the two member set $G = \{0, 1\}$ is shown in the adjoining table.

$\oplus$	0	1
0	0	1
1	1	0

Answer these question

- (i) Name identity element if it exists?
- (ii) What is the inverse of 1?
- (iii) Is the set G under the given operation a group.
- (iv) Abelian or non-abelian.

Solution:

Let $G = \{0, 1\}$

- (i) '0' is the identity element
- (ii) Inverse of 1 is 1
- (iii) Yes, it is a group because it satisfied all conditions for a group.

Because $1 + 0 = 0 + 1$

Q.2 The operation $\oplus$ as performed on the set $\{0, 1, 2, 3\}$ is shown in adjoining table. Show that the set is an abelian group.

Solution:

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Let $G = \{0, 1, 2, 3\}$

- (i) G is closed w.r.t. $\oplus$ because each element in the table belongs to G .
- (ii) $\oplus$ is associative in G because $\forall a, b, c \in G$ $(a \oplus b) \oplus c = a \oplus (b \oplus c)$
- (iii) '0' is the identity element in G . Because $\forall a \in G$ $a + 0 = a = 0 + a$.
- (iv) As inverse of each element of G exists in G , w.r.t. $\oplus$

inverse of 0 = 0

inverse of 1 = 3

inverse of 2 = 2

inverse of 3 = 1

- (v) G satisfies commutative property w.r.t. $\oplus$

i.e. $\forall a, b \in G$

$a \oplus b = b \oplus a$

$\Rightarrow$ 'G' is an Abelian group w.r.t. $\oplus$.



Q.3 For each of the following sets, determine whether or not the set forms a group w.r.t. indicated operation

- (i) The set of rational numbers w.r.t. '×'.
- (ii) The set of rational numbers w.r.t. '+'.
 - (iii) The set of positive rational numbers w.r.t. '×'.
 - (iv) The set of integers w.r.t. '+'.
 - (v) The set of integers w.r.t. '×'.

Solution:

(i) The set of rational numbers w.r.t. '×'.

The set of rational numbers w.r.t. multiplication is not a group because inverse of 0 does not exist.

(ii) The set of rational numbers w.r.t. '+'.

- (iii) The set of positive rational numbers w.r.t. '×'.
- (iv) The set of integers w.r.t. '+'.
 - (v) The set of integers w.r.t. '×'.

The set of rational numbers w.r.t. + is a group.

because

$$c-1 \quad \forall a, b \in Q$$

$$a + b \in Q$$

$$\Rightarrow Q \text{ is closed w.r.t. } +$$

$$c-2 \quad a, b, c \in Q$$

$$(a + b) + c = a + (b + c)$$

$$\Rightarrow '+' \text{ is associative in } Q$$

$$c-3 \quad '0' \text{ is the identity element in } Q$$

$$\text{such that } \forall a \in Q, a + 0 = 0 + a = a$$

$$c-4 \quad \text{Inverse of each element in } Q \text{ belongs to } Q.$$

$$\text{i.e. } \forall a \in Q \exists -a \in Q \text{ such that}$$

$$a + (-a) = 0 = (-a) + a$$

As all conditions for a group are satisfied so $(Q, +)$ is a group.

(iii) The set of positive rational numbers w.r.t. '×'

The set of positive rational numbers Q^+ is a group under multiplication. Because

$$c-1 \quad \forall a, b \in Q^+$$

$$a \cdot b \in Q^+$$

$$\Rightarrow Q^+ \text{ is closed under } '\cdot'$$

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$



$\Rightarrow$ '•' is associative in Q^+ .

c-3 $1 \in Q^+$ and 1 is an identity element in Q^+ .
such that $\forall a \in Q^+, a \cdot 1 = 1 \cdot a = a$

c-4 As $0 \notin Q^+$ and inverse of each element of Q^+ exist in Q^+ i.e.
 $\forall a \in Q^+, \frac{1}{a} \in Q^+$ such that

$$a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$$

As all the conditions for a group are satisfied so Q^+ is a group under multiplication.

(iv) **The set of integers w.r.t. '+'.**

The set of integers Z w.r.t. $+$ is a group because

c-1 $\forall a, b \in Z,$

'Z' is closed under $+$.

$$(a + b) + c = a + (b + c)$$

$\Rightarrow$ '+' is associative.

C-3 '0' is the identity element in Z .

$$\text{i.e. } \forall a \in Z$$

$$a + 0 = 0 + a = a$$

c-4 Inverse of each element in Z exists in Z .

i.e. $\forall a \in Z \exists -a \in Z$ such that

$$a + (-a) = (-a) + a = 0$$

$\Rightarrow (Z, +)$ is a group.

(v) **The set of integers w.r.t. 'x'.**

The set of integers is not a group under multiplication because inverse of 0 does not exist in Z .

Q.4 Show that the adjoining table represents the sum of the elements of the set $\{E, 0\}$. What is an identity element of this set? Show that this set is an abelian group.

$\oplus$	E	0
E	E	0
0	0	E

Solution:

$$\text{As } E + E = E$$

$$E + 0 = 0$$

$$0 + E = 0$$

$$0 + 0 = E$$

'E' is the identity element in $\{E, 0\}$ $\{E, 0\}$ is an abelian group under $+$

because



- c – 1 all elements in table belongs to $\{E, O\}$
 $\Rightarrow \{E, O\}$ is closed under '+'.
 c – 2 As associative law holds in $\{E, O\}$
 $\Rightarrow$ '+' is associative.
 c – 3 'E' is the identity element in $\{E, O\}$
 c – 4 inverse of E = E
 inverse of O = O
 $\Rightarrow$ inverse of each element in $\{E, O\}$ belongs to $\{E, O\}$
 c – 5 As
 $E \oplus O = O \oplus E$
 $\Rightarrow$ Commutative law holds under '+'.
 $\Rightarrow \{E, O\}$ is an abelian group.

Q.5 Show that the set $\{1, \omega, \omega^2\}$ when $\omega^3 = 1$ is an abelian group w.r.t. ordinary multiplication. (Gujranwala Board 2007)

Solution:

Let $G = \{1, \omega, \omega^2\}$

First we construct multiplication table

$\bullet$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

Now

- c – 1 As all elements in table belong to G.
 $\Rightarrow$ 'G' is closed under multiplication.
 c – 2 operation ' $\bullet$ ' is associative because
 $\forall a, b, c \in G, (a \cdot b) \cdot c = a \cdot (b \cdot c)$
 c – 3 $1 \in G$ is an identity element
 c – 4 inverse of 1 = 1
 inverse of $\omega = \omega^2$
 inverse of $\omega^2 = \omega$
 $\Rightarrow$ inverse of each element of G exist in G.
 c – 5 commutative law holds in G w.r.t. ' $\bullet$ '
 because $\forall a, b \in G, a \cdot b = b \cdot a$
 $\Rightarrow G = \{1, \omega, \omega^2\}$ is an abelian group under multiplication.



Q.6 If G is a group under the operation $*$ and $a, b \in G$, find the solution of equations: $a * x = b$, $x * a = b$. (Lahore Board 2010)

Solution:

Given

$$a * x = b$$

$$a^{-1} * (a * x) = a^{-1} * b$$

$$(a^{-1} * a) * x = a^{-1} * b \quad \text{by associative law}$$

$$e * x = a^{-1} * b$$

$$x = a^{-1} * b$$

Now

$$x * a = b$$

$$(x * a) * a^{-1} = b * a^{-1}$$

$$x * (a * a^{-1}) = b * a^{-1} \quad \text{by associative law}$$

$$x * e = b * a^{-1}$$

$$x = b * a^{-1}$$

which is required solution.

Q.7 Show that set consisting of elements of form $a + \sqrt{3}b$ (a, b being rational), is an abelian group w.r.t. addition. (Gujranwala Board, Lahore Board 2007)

Solution:

$$\text{Let } S = \{ a + \sqrt{3}b, a, b \in \mathbb{Q} \}$$

$$c-1 \quad \forall \quad a + \sqrt{3}b, c + \sqrt{3}d \in G$$

$$(a + \sqrt{3}b) + (c + \sqrt{3}d)$$

$$= a + \sqrt{3} + c + \sqrt{3}d = (a + c) + \sqrt{3}(b + d) \in G$$

$\Rightarrow$ G is closed w.r.t. addition.

c-2 '+' is associative in G because

$$\forall \quad a + \sqrt{3}b, c + \sqrt{3}d, e + \sqrt{3}f \in G$$

$$[(a + \sqrt{3}b) + (c + \sqrt{3}d)] + (e + \sqrt{3}f)$$

$$= (a + \sqrt{3}b) + [(c + \sqrt{3}d) + (e + \sqrt{3}f)]$$

c-3 $0 + \sqrt{3}0 \in G$ such that



$$\forall a + \sqrt{3} b \in G$$

$$(0 + \sqrt{3} 0) + (a + \sqrt{3} b)$$

$$= a + \sqrt{3} b$$

$$= (a + \sqrt{3} b) + (0 + \sqrt{3} 0)$$

identity element exists in G .

$$c-4 \quad \forall (a + \sqrt{3} b) \in G \quad \exists (-a - \sqrt{3} b) \in G$$

$$\text{such that } (a + \sqrt{3} b) + (-a - \sqrt{3} b) = 0 + \sqrt{3} 0$$

$\Rightarrow$ inverse of each element exists in G .

$$c-5 \quad \forall (a + \sqrt{3} b), (c + \sqrt{3} d) \in G$$

$$(a + \sqrt{3} b) + (c + \sqrt{3} d) = a + \sqrt{3} b + c + \sqrt{3} d$$

$$= a + c + \sqrt{3} (b + d)$$

$$= (c + \sqrt{3} d) + (a + \sqrt{3} b)$$

$\Rightarrow$ Commutative law holds in G .

$\Rightarrow$ G is an abelian group w.r.t. addition.

Q.8 Determine whether $(P(S), *)$, where $*$ stands for intersection is a semi-group, a monoid or neither. If it is monoid, specify its identity.

Solution:

$(P(S), *)$ is a monoid because

$$c-1 \quad \forall A, B \in P(S), A * B \in P(S)$$

$$\text{i.e. } A \cap B \in P(S)$$

$\Rightarrow$ $P(S)$ is closed under $*$

$$c-2 \quad \forall A, B, C \in P(S)$$

$$A * (B * C) = (A * B) * C$$

$$\text{i.e. } A \cap (B \cap C) = (A \cap B) \cap C$$

$\Rightarrow$ $*$ is associative in $P(S)$

$$c-3 \quad 'S' \text{ is the identity element in } P(S)$$

$$\text{i.e. } \forall A \in P(S)$$

$$A * S = A \cap S = A$$

$$\text{and } S * A = S \cap A = A$$

As inverse of each element in $P(S)$ does not exist in $P(S)$.

$\Rightarrow$ $(P(S), *)$ is a semi group and monoid.



Q.9 Complete the following table to obtain a semi-group under *.

*	a	b	c
a	c	a	b
b	a	b	c
c	—	—	a

Solution:

Let l, m are required elements.

then to obtain a semi group associative law must be satisfy

i.e $(a * a) * a = a * (a * a)$

$$c * a = a * c$$

$$l = b$$

also $(a * a) * b = a * (a * b)$

$$c * b = a * a$$

$$m = c$$

$\Rightarrow$ b, c are required elements.

Q.10 Prove that all 2×2 non-singular matrices over the real field form a non-abelian group under multiplication.

Solution:

$$\text{Let } G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}, a, b, c, d \in \mathbb{R}, ad - bc \neq 0 \right\}$$

c-1 Let $A, B \in G$

such that

$$A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$$

$$\begin{aligned} A \cdot B &= \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \cdot \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \\ &= \begin{bmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{bmatrix} \in G \end{aligned}$$

$\Rightarrow$ G is closed under multiplication.

c-2 ' $\bullet$ ' is associative in G .

because in matrices, $\forall A, B, C \in G$

$$(A \cdot B) \cdot C = A \cdot (B \cdot C)$$

c-3 $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in G$ which is an identity element in G such that $\forall A \in G$

$$A I_2 = A = I_2 A.$$

$\Rightarrow$ Identity element exists in G .

c-4 $\forall A \in G \exists A^{-1} \in G$ such that

$$A \cdot A^{-1} = I_2 = A^{-1} A$$



We can check it as

if

$$A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$$

then

$$\begin{aligned} A^{-1} &= \frac{\text{adj } A}{|A|} = \frac{1}{a_1 d_1 - b_1 c_1} \begin{bmatrix} d_1 & -b_1 \\ -c_1 & a_1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{d_1}{a_1 d_1 - b_1 c_1} & \frac{-b_1}{a_1 d_1 - b_1 c_1} \\ \frac{-c_1}{a_1 d_1 - b_1 c_1} & \frac{a_1}{a_1 d_1 - b_1 c_1} \end{bmatrix} \in G \end{aligned}$$

$\Rightarrow$ Inverse of each element in G exist in G .

c – 5 In matrices, we know that

$$\forall A, B \in G$$

$$A \cdot B \neq B \cdot A$$

$\Rightarrow$ Commutative law does not hold in G .

$\Rightarrow$ 'G' form a non-Abelian group under multiplication.

