

# CHAPTER

# 8

## MATHEMATICAL INDUCTION AND BINOMIAL THEOREM

### PRINCIPLE OF MATHEMATICAL INDUCTION (Lahore Board 2009)

The principle of mathematical induction is stated as follows:

$S(1)$  is true i.e.,  $S(n)$  is true for  $n = 1$  and  $S(k+1)$  is true whenever  $S(k)$  is true for any positive integer  $k$ , then  $S(n)$  is true for all positive integers.

### PROCEDURE

**Condition 1:** Substituting  $n = 1$ , show that the statement is true for  $n = 1$ .

**Condition 2:** Assuming that the statement is true for positive integer  $k$ , then show that it is true for the next higher integer.

## EXERCISE 8.1

**Q.1** Use mathematical induction to prove that the following formula for every positive integer  $n$ .

$$1 + 5 + 9 + \dots + (4n - 3) = n(2n - 1)$$

**Solution:**

Let  $S(n)$  be the given statement, that is

$$S(n): 1 + 5 + 9 + \dots + (4n - 3) = n(2n - 1)$$

**Condition 1**

When  $n = 1$ ,  $S(1)$  becomes

$$S(1): 4(1) - 3 = 1(2 - 1)$$

$$1 = 1$$

L.H.S. = R.H.S. Thus C-1 is satisfied.

**Condition 2**

Let us suppose that  $S(n)$  is true for any  $n = K \in \mathbb{N}$ , that is  $S(k)$ ;



$$1 + 5 + 9 + \dots + (4k - 3) = k(2k - 1) \quad \dots\dots\dots (1)$$

Adding  $[4(k + 1) - 3]$  on both sides of (1),

$$\begin{aligned} 1 + 5 + 9 + \dots + (4k - 3) + [4(k + 1) - 3] &= k(2k - 1) + [4(k + 1) - 3] \\ &= 2k^2 - k + 4k + 4 - 3 \\ &= 2k^2 + 3k + 1 \\ &= 2k^2 + 2k + k + 1 \\ &= 2k(k + 1) + 1(k + 1) \\ &= (k + 1)(2k + 1) \\ &= (k + 1)(2k + 2 - 1) \\ &= (k + 1)[2(k + 1) - 1] \end{aligned}$$

$$\Rightarrow n = k + 1$$

Thus  $S(k + 1)$  is true if  $S(k)$  is true. Therefore C-2 is satisfied.

Since both conditions are satisfied. Therefore  $S(n)$  is true for any positive integer.

**Q.2 Use mathematical induction to prove that the following formula for every positive integer  $n$ .**

$$1 + 3 + 5 + \dots + (2n - 1) = n^2 \quad (\text{Lahore Board 2005, 2008})$$

**Solution:**

Let  $S(n)$  be the given statement, that is

$$S(n): 1 + 3 + 5 + \dots + (2n - 1) = n^2$$

**Condition 1**

$$\text{When } n = 1$$

$$2 - 1 = 1$$

$$1 = 1$$

$$\text{L.H.S.} = \text{R.H.S.}$$

Therefore condition 1 is satisfied.

**Condition 2**

Let us suppose that  $S(n)$  is true for any  $n = K \in \mathbb{N}$ , that is

$$S(k): 1 + 3 + 5 + \dots + (2k - 1) = k^2 \quad \dots\dots\dots (1)$$

Adding  $[2(k + 1) - 1]$  on both sides



$$\begin{aligned}
 1 + 3 + 5 + \dots + (2k - 1) + [2(k + 1) - 1] &= k^2 + [2(k + 1) - 1] \\
 &= k^2 + 2k + 2 - 1 \\
 &= k^2 + 2k + 1 \\
 &= (k + 1)^2
 \end{aligned}$$

Thus  $S(k + 1)$  is true when  $S(k)$  is true. Therefore condition 2 is satisfied.

Since both conditions are satisfied, so  $S(n)$  is true for every positive integer.

**Q.3 Use mathematical induction to prove that the following formula for every positive integer  $n$ .**

$$1 + 4 + 7 + \dots + (3n - 2) = \frac{n(3n - 1)}{2}$$

**Solution:**

Let  $S(n)$  be the given statement, that is

$$S(n): 1 + 4 + 7 + \dots + (3n - 2) = \frac{n(3n - 1)}{2}$$

**Condition 1**

When  $n = 1$ ,  $S(1)$  becomes

$$1 = \frac{1(3 - 1)}{2}$$

$$1 = \frac{2}{2}$$

$$1 = 1$$

$$\text{L.H.S.} = \text{R.H.S.}$$

Therefore condition 1 is satisfied.

**Condition 2**

Let us suppose that  $S(n)$  is true for any  $n = K \in \mathbb{N}$ , that is

$$S(k): 1 + 4 + 7 + \dots + (3k - 2) = \frac{k(3k - 1)}{2} \quad \dots \dots \dots (1)$$

Adding  $[3(k + 1) - 2]$  on both sides of (1)

$$\begin{aligned}
 1 + 4 + 7 + \dots + (3k - 2) + [3(k + 1) - 2] &= \frac{k(3k - 1)}{2} + [3(k + 1) - 2] \\
 &= \frac{k(3k - 1)}{2} + 3k + 1
 \end{aligned}$$



$$\begin{aligned}
&= \frac{3k^2 - k + 6k + 2}{2} \\
&= \frac{3k^2 + 5k + 2}{2} \\
&= \frac{3k^2 + 3k + 2k + 2}{2} \\
&= \frac{3k(k+1) + 2(k+1)}{2} \\
&= \frac{(k+1)(3k+2)}{2} \\
&= \frac{(k+1)(3k+3-1)}{2} \\
&= \frac{(k+1)(3(k+1)-1)}{2}
\end{aligned}$$

$$\Rightarrow n = k + 1$$

Thus condition (2) is satisfied.

Hence  $S(n)$  is true for every +ve integer.

**Q.4 Use mathematical induction to prove that the following formula for every positive integer  $n$ .**

$$1 + 2 + 4 + \dots + 2^{n-1} = 2^n - 1$$

**Solution:**

Let  $S(n)$  be the given statement that is

$$S(n): 1 + 2 + 4 + \dots + 2^{n-1} = 2^n - 1$$

**Condition 1**

When  $n = 1$ ,  $S(1)$  becomes

$$2^{1-1} = 2^1 - 1$$

$$2^0 = 2 - 1$$

$$1 = 1$$

L.H.S. = R.H.S. therefore condition 1 is satisfied.

**Condition 2**

Let us assume that  $S(n)$  is true for any  $n = k \in \mathbb{N}$ , that is

$$S(k): 1 + 2 + 4 + \dots + 2^{k-1} = 2^k - 1 \quad \dots \dots \dots (1)$$



Adding  $2^{k+1-1}$  on both sides

$$\begin{aligned}
 1 + 2 + 4 + \dots + 2^{k-1} + 2^{k+1-1} &= 2^k - 1 + 2^{k+1-1} \\
 &= 2^k - 1 + 2^k \\
 &= 2 \cdot 2^k - 1 \quad (2^k + 2^k = 2^{k+1}) \\
 &= 2^{k+1} - 1
 \end{aligned}$$

Therefore condition 2 is satisfied. Both conditions are satisfied. Hence  $S(n)$  is true for every +ve integer.

**Q.5 Use mathematical induction to prove that the following formula for every positive integer  $n$ . (Lahore Board 2010)**

$$1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{n-1}} = 2 \left[ 1 - \frac{1}{2^n} \right]$$

**Solution:**

Let  $S(n)$  be the given statement, that is

$$S(n): 1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{n-1}} = 2 \left[ 1 - \frac{1}{2^n} \right]$$

**Condition 1**

When  $n = 1$ ;  $S(n)$  becomes

$$S(1): 1 = 2 \left[ 1 - \frac{1}{2^1} \right]$$

$$1 = 2 \times \frac{1}{2} = 1$$

L.H.S. = R.H.S. therefore condition 1 is satisfied.

**Condition 2**

Let us suppose that  $S(n)$  is true for any  $n = k \in \mathbb{N}$ , that is

$$1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{k-1}} = 2 \left[ 1 - \frac{1}{2^k} \right] \quad \dots\dots\dots (1)$$

Adding  $\frac{1}{2^{k+1-1}}$  on both sides of (1)

$$1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{k-1}} + \frac{1}{2^{k+1-1}} = 2 \left[ 1 - \frac{1}{2^k} \right] + \frac{1}{2^{k+1-1}}$$



$$\begin{aligned}
&= 2 \left[ \frac{2^k - 1}{2^k} \right] + \frac{1}{2^k} \\
&= \frac{2 \cdot 2^k - 2 + 1}{2^k} = \frac{2^{k+1} - 1}{2^k} \\
&= \frac{2 [2^{k+1} - 1]}{2 \cdot 2^k} \quad (\text{Multiplying \& dividing by 2}) \\
&= \frac{2 [2^{k+1} - 1]}{2^{k+1}} \\
&= 2 \left[ \frac{2^{k+1}}{2^{k+1}} - \frac{1}{2^{k+1}} \right] \\
&= 2 \left[ 1 - \frac{1}{2^{k+1}} \right]
\end{aligned}$$

Thus  $S(k+1)$  is true if  $S(k)$  is true. Therefore condition 2 is satisfied. Hence  $S(n)$  is true for every positive integer.

**Q.6 Use mathematical induction to prove that the following formula for every positive integer  $n$ .**

$$2 + 4 + 6 + \dots + 2n = n(n+1)$$

**Solution:**

Let  $S(n)$  be the given statement given, that is

$$S(n): 2 + 4 + 6 + \dots + 2n = n(n+1)$$

**Condition 1**

When  $n = 1$ ;  $S(n)$  becomes

$$S(1), 2 = 1(1+1)$$

$$2 = 2$$

$$\text{L.H.S.} = \text{R.H.S.}$$

Therefore condition 1 is satisfied.

**Condition 2**

Let us suppose that  $S(n)$  is true for  $n = k \in \mathbb{N}$  that is

$$S(k): 2 + 4 + 6 + \dots + 2k = k(k+1) \quad \dots \dots \dots (1)$$

Adding  $2(k+1)$  on both sides

$$\begin{aligned}
2 + 4 + 6 + \dots + 2k + 2(k+1) &= k(k+1) + 2(k+1) \\
&= k^2 + k + 2k + 2
\end{aligned}$$



$$\begin{aligned}
 &= k(k+1) + 2(k+1) \\
 &= (k+1)(k+2) \\
 &= \overline{(k+1)}(\overline{k+1+1})
 \end{aligned}$$

Thus  $S(k+1)$  is true if  $S(k)$  is true. Therefore condition 2 is satisfied.

Since both conditions are satisfied. Therefore  $S(n)$  is true for every +ve integer.

**Q.7 Use mathematical induction to prove that the following formula for every positive integer  $n$ . (Lahore Board 2006)**

$$2 + 6 + 18 + \dots + 2 \times 3^{n-1} = 3^n - 1$$

**Solution:**

Let  $S(n)$  be the given statement, that is

$$S(n): 2 + 6 + 18 + \dots + 2 \times 3^{n-1} = 3^n - 1$$

**Condition 1**

When  $n = 1$ ;  $S(n)$  becomes

$$S(1): 2 = 3 - 1$$

$$2 = 2$$

$\Rightarrow$  L.H.S. = R.H.S. therefore condition 1 is satisfied.

**Condition 2**

Let us suppose that  $S(n)$  is true for  $n = k \in \mathbb{N}$ , that is

$$S(k); 2 + 6 + 8 + \dots + 2 \times 3^{k-1} = 3^k - 1 \quad \dots \dots \dots (1)$$

Adding  $2 \times 3^{k+1-1}$  on both sides

$$\begin{aligned}
 2 + 6 + 8 + \dots + 2 \times 3^{k-1} + 2 \times 3^{k+1-1} &= 3^k - 1 + 2 \times 3^{k+1-1} \\
 &= 3^k + 2 \times 3^k - 1 \\
 &= 3^k(1 + 2) - 1 \\
 &= \overline{3^{k+1}} - 1
 \end{aligned}$$

Thus  $S(k+1)$  is true when  $S(k)$  is true.

Therefore condition 2 is satisfied. Since both conditions are satisfied. Therefore  $S(n)$  is true for every positive integer.

**Q.8 Use mathematical induction to prove that the following formula for every positive integer  $n$ . (Gujranwala Board 2003)**

$$1 \times 3 + 2 \times 5 + 3 \times 7 + \dots + n \times (2n + 1) = \frac{n(n+1)(4n+5)}{6}$$



**Solution:**

Let  $S(n)$  be the given statement, that is

$$S(n): 1 \times 3 + 2 \times 5 + 3 \times 7 + \dots + n(2n-1) = \frac{n(n+1)(4n+5)}{6}$$

**Condition 1**

When  $n = 1$ ,  $S(n)$  becomes

$$S(1): 1 \times 3 = \frac{1(2)(9)}{6}$$

$$3 = 3$$

$\Rightarrow$  L.H.S. = R.H.S.

Therefore condition 1 is satisfied.

**Condition 2**

Let us assume that  $S(n)$  is true for  $n = k \in \mathbb{N}$ , i.e.

$$S(k): 1 \times 3 + 2 \times 5 + 3 \times 7 + \dots + k(2k+1) = \frac{k(k+1)(4k+5)}{6} \dots\dots\dots (1)$$

Adding  $(k+1)(2(k+1)+1)$  on both sides of (1)

$$\begin{aligned} 1 \times 3 + 2 \times 5 + 3 \times 7 + \dots + k(2k+1) + (k+1)[2(k+1)+1] \\ &= \frac{k(k+1)(4k+5)}{6} + (k+1)(2k+3) \\ &= \frac{k(k+1)(4k+5) + 6(k+1)(2k+3)}{6} \\ &= \frac{(k+1)[k(4k+5) + 6(2k+3)]}{6} \\ &= \frac{(k+1)[4k^2 + 5k + 12k + 18]}{6} \\ &= \frac{(k+1)(4k^2 + 17k + 18)}{6} \\ &= \frac{(k+1)(4k^2 + 8k + 9k + 18)}{6} \\ &= \frac{(k+1)[4k(k+2) + 9(k+2)]}{6} \end{aligned}$$



$$\begin{aligned}
 &= \frac{(k+1)(k+2)(4k+9)}{6} \\
 &= \frac{(k+1)(k+2)(4k+4+5)}{6} \\
 &= \frac{(k+1)(k+2)(4(k+1)+5)}{6} \\
 &= \frac{(\overline{k+1})(\overline{k+1}+1)[4(\overline{k+1})+5]}{6}
 \end{aligned}$$

Thus  $S(k+1)$  is true when  $S(k)$  is true. Condition 2 is satisfied. Since both conditions are satisfied. Therefore  $S(n)$  is true for every +ve integer.

**Q.9 Use mathematical induction to prove that the following formula for every positive integer  $n$ .**

$$1 \times 2 + 2 \times 3 + 3 \times 4 + \dots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

**Solution:**

Let  $S(n)$  be the given statement, that is

$$S(n): 1 \times 2 + 2 \times 3 + 3 \times 4 + \dots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

**Condition 1**

When  $n = 1$ ;  $S(n)$  becomes

$$S(1): 1 \times 2 = \frac{1(2)(3)}{3}$$

$$2 = 2$$

$$\Rightarrow \text{L.H.S.} = \text{R.H.S.}$$

Hence condition 1 is satisfied.

**Condition 2**

Let us assume that  $S(n)$  is true for any  $n = k \in \mathbb{N}$ , i.e.

$$S(k): 1 \times 2 + 2 \times 3 + 3 \times 4 + \dots + k(k+1) = \frac{k(k+1)(k+2)}{3} \quad \dots\dots\dots (1)$$

Adding  $(k+1) \times (k+1+1)$  on both sides

$$\begin{aligned}
 &1 \times 2 + 2 \times 3 + 3 \times 4 + \dots + k(k+1) + (k+1)(k+1+1) \\
 &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)
 \end{aligned}$$



$$\begin{aligned}
 &= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3} \\
 &= \frac{(k+1)(k+2)[k+3]}{3} \quad (\text{taking common}) \\
 &= \frac{(k+1)(k+2)(k+3)}{3} \\
 &= \frac{(\overline{k+1})(\overline{k+1+1})(\overline{k+1+2})}{3}
 \end{aligned}$$

Thus  $S(k+1)$  is true if  $S(k)$  is true.

Therefore condition 2 is satisfied. Since both conditions are satisfied. Therefore,  $S(n)$  is true for every +ve integer.

**Q.10 Use mathematical induction to prove that the following formula for every positive integer  $n$ .**

$$1 \times 2 + 3 \times 4 + 5 \times 6 + \dots + (2n-1) \times 2n = \frac{n(n+1)(4n-1)}{3}$$

**Solution:**

Let  $S(n)$  be the given statement, that is

$$S(n): 1 \times 2 + 3 \times 4 + 5 \times 6 + \dots + (2n-1)(2n) = \frac{n(n+1)(4n-1)}{3}$$

**Condition 1**

When  $n = 1$ ,  $S(n)$  becomes

$$S(1): 1 \times 2 = \frac{1(1+1)(4-1)}{3}$$

$$2 = \frac{2 \times 3}{3}$$

L.H.S. = R.H.S. therefore condition 1 is satisfied.

**Condition 2**

Let us suppose that  $S(n)$  is true for  $n = k \in \mathbb{N}$  that is

$$S(k): 1 \times 2 + 3 \times 4 + 5 \times 6 + \dots + (2k-1) \times 2k = \frac{k(k+1)(4k-1)}{3} \quad \dots (1)$$

Adding  $[2(k+1)-1] \times 2(k+1)$  on both sides of (1)

$$1 \times 2 + 3 \times 4 + 5 \times 6 + \dots + (2k-1) \times 2k + [2(k+1)-1] \times 2(k+1)$$



$$\begin{aligned}
&= \frac{k(k+1)(4k-1)}{3} + [2(k+1)-1]2(k+1) \\
&= \frac{k(k+1)(4k-1)}{3} + (2k+1) \times 2(k+1) \\
&= \frac{k(k+1)(4k-1) + 6(2k+1)(k+1)}{3} \\
&= \frac{(k+1)[4k^2 - k + 12k + 6]}{3} \\
&= \frac{(k+1)(4k^2 + 11k + 6)}{3} \\
&= \frac{(k+1)(4k^2 + 8k + 3k + 6)}{6} \\
&= \frac{(k+1)(4k+3)(k+2)}{6} \\
&= \frac{(k+1)(k+1+1)(4k+4-1)}{6} \\
&= \frac{(\overline{k+1})(\overline{k+1+1})(\overline{4k+1-1})}{6}
\end{aligned}$$

Thus  $S(k+1)$  is true if  $S(k)$  is true.

Therefore condition 2 is satisfied. Since both conditions are satisfied.

Therefore  $S(n)$  is true for every +ve integer.

**Q.11 Use mathematical induction to prove that the following formula for every positive integer  $n$ .**

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1} = \frac{n+1-1}{n+1} = \frac{n}{n+1}$$

**Solution:**

Let  $S(n)$  be the given statement, that is

$$S(n): \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

**Condition 1**

When  $n = 1$ ;  $S(n)$  becomes

$$S(1): \frac{1}{1 \times 2} = \frac{1}{1+1}$$



$$\frac{1}{2} = \frac{1}{2}$$

$\Rightarrow$  L.H.S. = R.H.S. therefore condition 1 is satisfied.

### Condition 2

Let us suppose that  $S(n)$  is true for  $n = k \in \mathbb{N}$  that is

$$S(k); \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1} \quad \dots\dots\dots (1)$$

Adding  $\frac{1}{(k+1)(k+1+1)}$  on both sides

$$\begin{aligned} \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+1+1)} \\ &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k(k+2) + 1}{(k+1)(k+2)} \\ &= \frac{k^2 + 2k + 1}{(k+1)(k+2)} \\ &= \frac{k^2 + 2k + 1}{(k+1)(k+2)} \\ &= \frac{(k+1)^2}{(k+1)(k+2)} \\ &= \frac{(k+1)}{k+1+1} \end{aligned}$$

Thus  $S(k+1)$  is true when  $S(k)$  is true.

Therefore condition 2 is satisfied. Since both conditions are satisfied.

Therefore  $S(n)$  is true for every +ve integer.

**Q.12 Use mathematical induction to prove that the following formula for every positive integer  $n$ .**

$$\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \dots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

**Solution:**

Let  $S(n)$  be the given statement, that is

$$S(n): \frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \dots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$



**Condition 1**

When  $n = 1$ ,  $S(n)$  becomes

$$S(1): \frac{1}{(1)(3)} = \frac{1}{2+1}$$

$$\frac{1}{3} = \frac{1}{3}$$

$\Rightarrow$  L.H.S. = R.H.S.

Therefore condition 1 is satisfied.

**Condition 2**

Let us suppose that  $S(n)$  is true for  $n = k \in \mathbb{N}$  that is

$$\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \dots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1} \quad \dots\dots\dots (1)$$

Adding  $\frac{1}{[2(k+1)-1][2(k+1)+1]}$  on both sides of (1)

$$\begin{aligned} \frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \dots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{[2(k+1)-1][2(k+1)+1]} \\ &= \frac{k}{(2k+1)} + \frac{1}{(2k+1)(2k+3)} \\ &= \frac{k(2k+3) + 1}{(2k+1)(2k+3)} \\ &= \frac{2k^2 + 3k + 1}{(2k+1)(2k+3)} \\ &= \frac{2k^2 + 2k + k + 1}{(2k+1)(2k+3)} \\ &= \frac{2k(k+1) + 1(k+1)}{(2k+1)(2k+3)} \\ &= \frac{(k+1)(2k+1)}{(2k+1)(2k+3)} \\ &= \frac{k+1}{2k+2+1} = \frac{(k+1)}{2(k+1)+1} \end{aligned}$$

Thus  $S(k+1)$  is true, when  $S(k)$  is true. Therefore condition 2 is satisfied.

Hence  $S(n)$  is true for every +ve integer.



**Q.13** Use mathematical induction to prove that the following formula for every positive integer  $n$ .

$$\frac{1}{2 \times 5} + \frac{1}{5 \times 8} + \frac{1}{8 \times 11} + \dots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{2(3n+2)}$$

**Solution:**

Let  $S(n)$  be the given statement, that is

$$S(n): \frac{1}{2 \times 5} + \frac{1}{5 \times 8} + \frac{1}{8 \times 11} + \dots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{2(3n+2)}$$

**Condition 1**

When  $n = 1$ ,  $S(n)$  becomes

$$S(1): \frac{1}{2 \times 5} = \frac{1}{2(3+2)}$$

$$\frac{1}{10} = \frac{1}{10}$$

$\Rightarrow$  L.H.S. = R.H.S.

Therefore condition 1 is satisfied.

**Condition 2**

Let us suppose that  $S(n)$  is true for  $n = k \in \mathbb{N}$ , i.e.

$$S(k): \frac{1}{2 \times 5} + \frac{1}{5 \times 8} + \frac{1}{8 \times 11} + \dots + \frac{1}{(3k-1)(3k+2)} = \frac{k}{2(3k+2)} \quad \dots\dots\dots (1)$$

Adding  $\frac{1}{[3(k+1)-1][3(k+1)+2]}$  on both sides

$$\begin{aligned} \frac{1}{2 \times 5} + \frac{1}{5 \times 8} + \frac{1}{8 \times 11} + \dots + \frac{1}{(3k-1)(3k+2)} + \frac{1}{[3(k+1)-1][3(k+1)+2]} \\ = \frac{k}{2(3k+2)} + \frac{1}{[3(k+1)-1][3(k+1)+2]} \\ = \frac{k}{2(3k+2)} + \frac{1}{(3k+2)(3k+5)} \\ = \frac{k(3k+5) + 2}{2(3k+2)(3k+5)} \end{aligned}$$



$$\begin{aligned}
 &= \frac{3k^2 + 5k + 2}{2(3k+2)(3k+5)} \\
 &= \frac{3k^2 + 3k + 2k + 2}{2(3k+2)(3k+5)} \\
 &= \frac{3k(k+1) + 2(k+1)}{2(3k+2)(3k+5)} \\
 &= \frac{(k+1)(3k+2)}{2(3k+2)(3k+5)} \\
 &= \frac{k+1}{2(3k+5)} = \frac{k+1}{2(3k+3+2)} \\
 &= \frac{(k+1)}{2[3(k+1)+2]}
 \end{aligned}$$

Thus  $S(k+1)$  is true, when  $S(k)$  is true.

Therefore condition 2 is satisfied. Thus  $S(n)$  is true for every +ve integer.

**Q.14 Use mathematical induction to prove that the following formula for every positive integer  $n$ .**

$$r + r^2 + r^3 + \dots + r^n = \frac{r(1-r^n)}{1-r}$$

**Solution:**

Let  $S(n)$  be the given statement, that is

$$S(n): r + r^2 + r^3 + \dots + r^n = \frac{r(1-r^n)}{1-r}$$

**Condition 1**

When  $n = 1$ ,  $S(n)$  becomes

$$S(1): r = \frac{r(1-r)}{(1-r)}$$

$$r = r$$

$$\Rightarrow \text{L.H.S.} = \text{R.H.S.}$$

Therefore condition 1 is satisfied.

**Condition 2**

Let us suppose that  $S(n)$  is true for any  $n = k \in \mathbb{N}$ , that is

$$S(k): r + r^2 + r^3 + \dots + r^k = \frac{r(1-r^k)}{1-r} \quad \dots\dots\dots (1)$$



Adding both sides by  $r^{k+1}$

$$\begin{aligned}
 r + r^2 + r^3 + \dots + r^k + r^{k+1} &= \frac{r(1-r^k)}{(1-r)} + r^{k+1} \\
 &= \frac{r(1-r^k) + (1-r)r^{k+1}}{(1-r)} \\
 &= \frac{r - r^{k+1} + r^{k+1} - r^{k+1+1}}{1-r} \\
 &= \frac{r - r^{k+1+1}}{1-r} \\
 &= \frac{r(1-r^{k+1})}{1-r}
 \end{aligned}$$

Thus  $S(k+1)$  is true when  $S(k)$  is true.

Therefore condition 2 is satisfied.

Hence  $S(n)$  is true for every +ve integer.

**Q.15 Use mathematical induction to prove that the following formula for every positive integer  $n$ .**

$$a + (a + d) + (a + 2d) + \dots + [a + (n-1)d] = \frac{n}{2} [2a + (n-1)d]$$

**Solution:**

Let  $S(n)$  be the given statement that is

$$S(n): a + (a + d) + (a + 2d) + \dots + [a + (n-1)d] = \frac{n}{2} [2a + (n-1)d]$$

**Condition 1**

When  $n = 1$ ,  $S(n)$  becomes

$$S(1): a = \frac{1}{2} [2a + (1-1)d] = \frac{1}{2} (2a)$$

$$a = a$$

$$\Rightarrow \text{L.H.S.} = \text{R.H.S.}$$

Therefore condition 1 is satisfied.

**Condition 2**

Let us suppose that  $S(n)$  is true for any  $n = k \in \mathbb{N}$ , i. e.

$$S(k): a + (a + d) + (a + 2d) + \dots + [a + (k-1)d] = \frac{k}{2} [2a + (k-1)d] \dots \dots \dots (1)$$



Adding both sides by  $[a + (k + 1 - 1) d]$

$$\begin{aligned}
 & a + (a + d) + (a + 2d) + \dots + [a + (k - 1) d] + [a + (k + 1 - 1) d] \\
 &= \frac{k}{2} [2a + (k - 1) d] + [a + (k + 1 - 1) d] \\
 &= \frac{k}{2} [2a + (k - 1) d] + [a + kd] \\
 &= \frac{k [2a + kd - d] + 2 [a + kd]}{2} \\
 &= \frac{2ak + k^2 d - kd + 2a + 2kd}{2} \\
 &= \frac{2ak + 2a + k^2 d + kd}{2} \\
 &= \frac{2a (k + 1) + kd (k + 1)}{2} \\
 &= \frac{(k + 1) (2a + kd)}{2} \\
 &= \frac{(k + 1)}{2} [2a + (k + 1 - 1) d]
 \end{aligned}$$

Thus  $S(k + 1)$  is true when  $S(k)$  is true.

Therefore condition 2 is satisfied.

Hence  $S(n)$  is true for every +ve integer.

**Q.16 Use mathematical induction to prove that the following formula for every positive integer  $n$ .**

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \dots + n \cdot n! = (n + 1)! - 1.$$

**Solution:**

Let  $S(n)$  be the given statement, that is

$$S(n): 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \dots + n \cdot n! = (n + 1)! - 1$$

**Condition 1**

When  $n = 1$ ,  $S(n)$  becomes

$$S(1): 1 \cdot 1! = (1 + 1)! - 1$$

$$1 = 2! - 1$$

$$1 = 2 - 1$$

$$1 = 1$$

$$\Rightarrow \text{L.H.S.} = \text{R.H.S.}$$

Thus condition 1 is satisfied.



**Condition 2**

Let us assume that  $S(n)$  is true for  $n = k \in \mathbb{N}$ , i.e.

$$S(k): 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \dots + k \cdot k! = (k+1)! - 1 \quad \dots\dots\dots (1)$$

Adding  $(k+1)(k+1)!$  on both sides of (1)

$$\begin{aligned} 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \dots + k \cdot k! + (k+1)(k+1)! \\ &= (k+1)! - 1 + (k+1)(k+1)! \\ &= (k+1)! + (k+1)(k+1)! - 1 \\ &= (k+1)! [1 + k + 1] - 1 \\ &= (k+1)! (k+2) - 1 \\ &= (k+2)(k+1)! - 1 \\ &= (k+2)! - 1 \\ &= (\overline{k+1+1}) - 1 \end{aligned}$$

Thus  $S(k+1)$  is true, when  $S(k)$  is true.

Therefore condition 2 is satisfied. Since both conditions are satisfied.

Therefore,  $S(n)$  is true for every +ve integer.

**Q.17 Use mathematical induction to prove that the following formula for every positive integer  $n$ .**

$$a_n = a_1 + (n-1)d \text{ when } a_1, a_1 + d, a_1 + 2d, \dots \text{ forms A.P.}$$

**Solution:**

Let  $S(n)$  be the given statement, that is

$$S(n): a_n = a_1 + (n-1)d$$

**Condition 1**

When  $n = 1$ ,  $S(n)$  becomes

$$S(1): a_1 = a_1 + (1-1)d$$

$$a_1 = a_1$$

$$\Rightarrow \text{L.H.S.} = \text{R.H.S.}$$

Therefore condition 1 is satisfied.



**Condition 2**

Let us suppose that  $S(n)$  is true for  $n = k \in \mathbb{N}$  that is

$$S(k): a_k = a_1 + (k-1)d \quad \dots\dots\dots (1)$$

Adding “d” on both sides

$$a_{k+d} = a_1 + (k-1)d + d$$

$$a_{k+1} = a_1 + d[k-1+1]$$

$$a_{\overline{k+1}} = a_1 + (\overline{k+1} - 1)d$$

Thus  $S(k+1)$  is true when  $S(k)$  is true.

Therefore condition 2 is satisfied.

Since both conditions are satisfied. Therefore  $S(n)$  is true for every +ve integer.

**Q.18 Use mathematical induction to prove that the following formula for every positive integer n.**

$$a_n = a_1 r^{n-1} \quad \text{when } a_1, a_1 r, a_1 r^2, \dots\dots\dots \text{ from a G. P}$$

**Solution:**

Let  $S(n)$  be the given statement, that is

$$S(n): a_n = a_1 r^{n-1}$$

**Condition 1**

When  $n = 1$ , then  $S(n)$  becomes

$$S(1): a_1 = a_1 r^{1-1}$$

$$a_1 = a_1$$

$$\Rightarrow \text{L.H.S.} = \text{R.H.S.}$$

Therefore condition 1 is satisfied.

**Condition 2**

Let us suppose that  $S(n)$  is true for  $n = k \in \mathbb{N}$ , i.e.

$$S(k): a_k = a_1 r^{k-1} \quad \dots\dots\dots (1)$$

Multiplying both sides by  $r$

$$a_k \times r = a_1 r^{k-1} \times r$$

$$a_{k+1} = a_1 r^{k-1+1}$$

$$a_{\overline{k+1}} = a_1 r^{\overline{k+1}-1}$$



Thus  $S(k+1)$  is true when  $S(k)$  is true.

Therefore condition 2 is satisfied.

Since both conditions are true.

Therefore  $S(n)$  is true for every +ve integer.

**Q.19 Use mathematical induction to prove that the following formula for every positive integer  $n$ .**

$$1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \frac{n(4n^2-1)}{3}$$

**Solution:**

Let  $S(n)$  be the given statement, that is

$$S(n): 1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \frac{n(4n^2-1)}{3}$$

**Condition 1**

When  $n = 1$ ,  $S(n)$  becomes

$$S(1): 1^2 = \frac{1(4-1)}{3}$$

$$1 = 1$$

$\Rightarrow$  L.H.S. = R.H.S.

Therefore condition 1 is satisfied.

**Condition 2**

Let us suppose that  $S(n)$  is true for  $n = k \in \mathbb{N}$ , i.e.

$$S(k): 1^2 + 3^2 + 5^2 + \dots + (2k-1)^2 = \frac{k(4k^2-1)}{3} \quad \dots\dots\dots (1)$$

Adding  $[2(k+1)-1]^2$  on both sides

$$\begin{aligned} 1^2 + 3^2 + 5^2 + \dots + (2k-1)^2 + [2(k+1)-1]^2 &= \frac{k(4k^2-1)}{3} + (2k+1)^2 \\ &= \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2 \\ &= \frac{k(2k-1)(2k+1) + 3(2k+1)^2}{3} \\ &= \frac{(2k+1)[k(2k-1) + 3(2k+1)]}{3} \end{aligned}$$



$$\begin{aligned}
 &= \frac{(2k+1)(2k-1)(k+3)}{3} \\
 &= \frac{(2k+1)(2k^2+6k-k-3)}{3} \\
 &= \frac{(2k+1)(2k+3)(k+1)}{3} \\
 &= \frac{(k+1)[(2k+1)(2k+3)]}{3} \\
 &= \frac{(k+1)[4k^2+6k+2k+3]}{3} \\
 &= \frac{(k+1)[4k^2+8k+4-1]}{3} \\
 &= \frac{(k+1)[4(k^2+2k+1)-1]}{3} \\
 &= \frac{(k+1)[4(k+1)^2-1]}{3}
 \end{aligned}$$

Thus  $S(k+1)$  is true, when  $S(k)$  is true.

Therefore condition 2 is satisfied.

Since both the conditions are satisfied.

Therefore  $S(n)$  is true for every +ve integer.

**Q.20 Use mathematical induction to prove that the following formula for every positive integer  $n$ . (Lahore Board 2010)**

$$\binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \dots + \binom{n+2}{3} = \binom{n+3}{4}$$

**Solution:**

Let  $S(n)$  be the given statement, therefore

$$S(n): \binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \dots + \binom{n+2}{3} = \binom{n+3}{4}$$

**Condition 1**

When  $n = 1$ ,  $S(n)$  becomes

$$\binom{3}{3} = \binom{1+3}{4}$$

$$1 = 1$$



$$\Rightarrow \text{L.H.S.} = \text{R.H.S.}$$

Therefore condition 1 is satisfied.

### Condition 2

Let us suppose that  $S(n)$  is true for  $n = k \in \mathbb{N}$ , i.e.

$$S(k): \binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \dots + \binom{k+2}{3} = \binom{k+3}{4} \quad \dots\dots\dots (1)$$

Adding  $\binom{k+2+1}{3}$  on both sides of (1) gives

$$\begin{aligned} \binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \dots + \binom{k+2}{3} + \binom{k+2+1}{3} &= \binom{k+3}{4} + \binom{k+2+1}{4} \\ &= \binom{k+3+1}{4} \quad \left( \begin{array}{c} n \quad n \quad n+1 \\ c + c = c \\ r \quad r-1 \quad r \end{array} \right) \\ &= \binom{k+1+3}{4} \end{aligned}$$

Thus  $S(k+1)$  is true when  $S(k)$  is true.

Therefore condition 2 is satisfied. Since both conditions are satisfied.

Therefore  $S(n)$  is true for every +ve integer.

**Q.21 Use mathematical induction to prove that the following formula for every positive integer  $n$ . (Lahore Board 2011)**

- (i)  $n^2 + n$  is divisible by 2
- (ii)  $5^n - 2^n$  is divisible by 3
- (iii)  $5^n - 1$  is divisible by 4
- (iv)  $8 \times 10^n - 2$  is divisible by 6
- (v)  $n^3 - n$  is divisible by 6

**Solution:**

- (i)  $n^2 + n$  is divisible by 2

Let  $S(n)$  be the given statement > that is

$S(n); n^2 + n$  is divisible by 2

### Condition 1

When  $n = 1$ ,  $S(n)$  becomes

$$S(1): 1^2 + 1 = 1 + 1 = 2$$



Clearly 2 is divisible by 2

Therefore condition 1 is satisfied.

### Condition 2

Let us suppose that  $S(n)$  is true for  $n = k \in \mathbb{N}$ , i.e.

$$S(k): k^2 + k \text{ is divisible by 2} \quad \dots\dots\dots (1)$$

We want to prove that  $S(k+1)$  is also divisible by 2

For  $n = k+1$   $S(n)$  becomes

$$\begin{aligned} S(k+1) &= (k+1)^2 + (k+1) \\ &= k^2 + 1 + 2k + k + 1 \\ &= k^2 + k + 2k + 2 \\ &= (k^2 + k) + 2(k+1) \end{aligned}$$

$(k^2 + k)$  is divisible by 2 by expression (1) and  $2(k+1)$  is also divisible by 2. Thus  $S(k+1)$  is divisible by 2.

Therefore condition 2 is satisfied. Since both conditions are satisfied.

Therefore  $S(n)$  is divisible by 2 for all +ve integers.

### (ii) $5^n - 2^n$ is divisible by 3 (Gujranawala Board, 2006)

Let  $S(n)$  be the given statement, i.e.

$$S(n): 5^n - 2^n \text{ is divisible by 3}$$

### Condition 1

When  $n = 1$ ;  $S(n)$  becomes

$$S(1); 5 - 2 = 3, \text{ clearly 3 is divisible by 3}$$

Therefore condition 1 is satisfied.

### Condition 2

Let us suppose that  $S(n)$  is true for  $n = k \in \mathbb{N}$ , that is

$$S(k); 5^k - 2^k \text{ is divisible by 3 (1)}$$

We want to prove that  $S(k+1)$  is also divisible by 3

For  $n = k+1$   $S(n)$  becomes

$$\begin{aligned} S(k+1); 5^{k+1} - 2^{k+1} &= 5^k \cdot 5 - 2^k \cdot 2 \\ &= (3+2) 5^k - 2 \cdot 2^k \\ &= 3 \cdot 5^k + 2 \cdot 5^k - 2 \cdot 2^k \\ &= 3 \cdot 5^k + 2(5^k - 2^k) \end{aligned}$$

The first term is clearly divisible by 3. The  $2^{nd}$  term also divisible by 3 by (1). Thus the whole term is divisible by 3. Therefore condition 2 is satisfied. Thus  $S(n)$  is divisible by every +ve integer.



**(iii)  $5^n - 1$  is divisible by 4**

Let  $S(n)$  be the given statement, i.e.

$$S(n) = 5^n - 1 \text{ is divisible by 4}$$

**Condition 1**

When  $n = 1$ , then  $S(n)$  becomes

$$S(1) = 5 - 1 = 4$$

Clearly 4 is divisible by 4. Therefore condition 1 is satisfied.

**Condition 2**

Let us suppose that  $S(n)$  is true for  $n = k \in \mathbb{N}$ , i.e

$$S(k) = 5^k - 1 \text{ is divisible by 4} \quad \dots\dots\dots (1)$$

We want to prove that  $S(k+1)$  is also divisible by 4

For  $n = k+1$   $S(n)$  becomes

$$\begin{aligned} S(k+1); 5^{k+1} - 1 &= 5^k \cdot 5 - 1 \\ &= (4 + 1) 5^k - 1 \\ &= 4 \cdot 5^k + (5^k - 1) \end{aligned}$$

The first term is clearly divisible by 4. The second term is also divisible by 4 by (1).

Thus  $S(k+1)$  is divisible by 4. Therefore condition 2 is satisfied.

Thus  $S(n)$  is divisible for all +ve integral values of  $n$ .

**(iv)  $8 \times 10^n - 2$  is divisible by 6 (Lahore Board 2010)**

Let  $S(n)$  be the given statement, that is

$$S(n); 8 \times 10^n - 2 \text{ is divisible by 6}$$

**Condition 1**

When  $n = 1$ ;  $S(n)$  becomes

$$S(1); 8 \times 10 - 2 = 80 - 2 = 78 \text{ clearly divisibly by 6}$$

**Condition 2**

Let us suppose that  $S(n)$  is true for  $n = k \in \mathbb{N}$  that is



$$S(k); 8 \times 10^k - 2 \text{ is divisible by } 6 \quad \dots\dots\dots (1)$$

We want to prove that  $S(k+1)$  is also divisible by 6.

For  $n = k+1$   $S(n)$  becomes

$$\begin{aligned} S(k+1); 8 \times 10^{k+1} - 2 &= 8 \times 10^k \cdot 10 - 2 \\ &= 8 \times 10^k \cdot 10 - 2 \\ &= 80 \times 10^k - 2 \\ &= (72 + 8) 10^k - 2 \\ &= 72 \times 10^k + (8 \times 10^k - 2) \end{aligned}$$

The first term is clearly divisible by 6. The second term is also divisible by 6 by (1).

Thus  $S(k+1)$  is divisible by 6. Therefore condition 2 is satisfied.

Hence  $S(n)$  is divisible by 6 for all +ve integers.

**(v)  $n^3 - n$  is divisible by 6**

Let  $S(n)$  be the given statement, that is

$$S(n); n^3 - n \text{ is divisible by } 6$$

**Condition 1**

When  $n = 1$ ,  $S(n)$  becomes

$$S(1) = 1 - 1 = 0 \text{ which is divisible by } 6. \text{ Therefore condition 1 is satisfied.}$$

**Condition 2**

Let us suppose that  $S(n)$  is true for  $n = k \in \mathbb{N}$ , i.e.

$$S(k) = k^3 - k \text{ is divisible by } 6 \quad \dots\dots\dots (1)$$

We want to prove that  $S(k+1)$  is divisible by 6

For  $n = k+1$   $S(n)$  becomes

$$\begin{aligned} S(k+1) &= (k+1)^3 - (k+1) \\ &= (k+1) [(k+1)^2 - 1] \\ &= (k+1) [k^2 + 2k + 1 - 1] \\ &= (k+1) (k^2 + 2k) \\ &= (k+1) k (k+2) \\ &= k (k+1) (k+2) \end{aligned}$$

Since the product of three consecutive terms is divisible by 6. Thus  $S(k+1)$  is divisible by 6. Therefore condition 2 is satisfied. Hence  $S(n)$  is divisible by 6 for all +ve integral values of  $n$ .



**Q.22** Use mathematical induction to prove that the following formula for every positive integer  $n$ .  
(Lahore Board 2005)

$$\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots + \frac{1}{3^n} = \frac{1}{2} \left[ 1 - \frac{1}{3^n} \right]$$

**Solution:**

Let  $S(n)$  be the given statement, i.e.

$$S(n): \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots + \frac{1}{3^n} = \frac{1}{2} \left[ 1 - \frac{1}{3^n} \right]$$

**Condition 1**

When  $n = 1$ ,  $S(n)$  becomes

$$S(1): \frac{1}{3} = \frac{1}{2} \left[ 1 - \frac{1}{3} \right]$$

$$\frac{1}{3} = \frac{1}{2} \left( \frac{2}{3} \right)$$

$$\frac{1}{3} = \frac{1}{3}$$

$\Rightarrow$  L.H.S. = R.H.S.

Therefore condition 1 is satisfied.

**Condition 2**

Let us suppose that  $S(n)$  is true for  $n = k \in \mathbb{N}$ , i.e.

$$S(k): \frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^k} = \frac{1}{2} \left[ 1 - \frac{1}{3^k} \right] \quad \dots \dots \dots (1)$$

Adding  $\frac{1}{3^{k+1}}$  on both sides of (1)

$$\begin{aligned} \frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^k} + \frac{1}{3^{k+1}} &= \frac{1}{2} \left[ 1 - \frac{1}{3^k} \right] + \frac{1}{3^{k+1}} \\ &= \frac{1}{2} \left[ \frac{3^k - 1}{3^k} \right] + \frac{1}{3^{k+1}} \\ &= \frac{3[3^k - 1] + 2[1]}{2 \cdot 3^{k+1}} \\ &= \frac{3^{k+1} - 3 + 2}{2 \cdot 3^{k+1}} \end{aligned}$$



$$\begin{aligned}
 &= \frac{1}{2} \left[ \frac{3^{k+1} - 1}{3^{k+1}} \right] \\
 &= \frac{1}{2} \left[ 1 - \frac{1}{3^{k+1}} \right]
 \end{aligned}$$

Thus  $S(k+1)$  is true, when  $S(k)$  is true.

Therefore condition 2 is satisfied.

Since both conditions are satisfied. Therefore  $S(n)$  is true for every +ve integral values of  $n$ .

**Q.23 Use mathematical induction to prove that the following formula for every positive integer  $n$ .**

$$1^2 - 2^2 + 3^2 - 4^2 + \dots + (-1)^{n-1} \cdot n^2 = \frac{(-1)^{n-1} \cdot n(n+1)}{2}$$

**Solution:**

Let  $S(n)$  be the given statement, that is

$$S(n): 1^2 - 2^2 + 3^2 - 4^2 + \dots + (-1)^{n-1} \cdot n^2 = \frac{(-1)^{n-1} \cdot n(n+1)}{2}$$

**Condition 1**

When  $n = 1$ ,  $S(n)$  becomes

$$S(1): 1^2 = \frac{(-1)^{1-1} \cdot 1(1+1)}{2}$$

$$1 = 1$$

Since L.H.S. = R.H.S.

Therefore condition 1 is satisfied.

**Condition 2**

Let us suppose that  $S(n)$  be true for  $n = k \in \mathbb{N}$ , i.e.

$$S(k): 1^2 - 2^2 + 3^2 - 4^2 + \dots + (-1)^{k-1} \cdot k^2 = \frac{(-1)^{k-1} \cdot k(k+1)}{2} \quad \dots\dots\dots (1)$$

Adding  $(-1)^{k+1-1} (k+1)^2$  on both sides of equation (1)

$$\begin{aligned}
 &1^2 - 2^2 + 3^2 - 4^2 + \dots + (-1)^{k-1} k^2 + (-1)^{k+1-1} (k+1)^2 \\
 &= \frac{(-1)^{k-1} k(k+1)}{2} + (-1)^k (k+1)^2 \\
 &= \frac{(-1)^k (-1)^{-1} k(k+1) + 2(-1)^k (k+1)^2}{2}
 \end{aligned}$$



$$\begin{aligned}
 &= \frac{(-1)^k (k+1) [-k+2k+2]}{2} \\
 &= \frac{(-1)^k (k+1) (k+2)}{2} \\
 &= \frac{(-1)^{k+1-1} (\overline{k+1}) (\overline{k+1}+1)}{2}
 \end{aligned}$$

Thus  $S(k+1)$  is true if  $S(k)$  is true.

Therefore condition 2 is satisfied.

Hence  $S(n)$  is true for all positive integer  $n$ .

**Q.24 Use mathematical induction to prove that the following formula for every positive integer  $n$ .**

$$1^3 + 3^3 + 5^3 + \dots + (2n-1)^3 = n^2 [2n^2 - 1]$$

**Solution:**

Let  $S(n)$  be the given statement, that is

$$S(n): 1^3 + 3^3 + 5^3 + \dots + (2n-1)^3 = n^2 [2n^2 - 1]$$

**Condition 1**

When  $n = 1$ ,  $S(n)$  becomes

$$S(1): 1^3 = 1 [2 - 1]$$

$$1 = 1$$

$\Rightarrow$  L.H.S. = R.H.S.

Therefore condition 1 is satisfied.

**Condition 2**

Let us suppose that  $S(n)$  is true for  $n = k \in \mathbb{N}$  that is

$$1^3 + 3^3 + 5^3 + \dots + (2k-1)^3 = k^2 [2k^2 - 1] \quad \dots\dots\dots (1)$$

Adding  $[2(k+1)-1]^3$  on both sides

$$\begin{aligned}
 &1^3 + 3^3 + 5^3 + \dots + (2k-1)^3 + [2(k+1)-1]^3 \\
 &= k^2 [2k^2 - 1] + [2k+1]^3 \\
 &= 2k^4 - k^2 + 8k^3 + 1 + 12k^2 + 6k \\
 &= 2k^4 + 8k^3 + 11k^2 + 6k + 1 \\
 &= 2k^4 + 2k^3 + 6k^3 + 6k^2 + 5k^2 + 5k + k + 1
 \end{aligned}$$



$$\begin{aligned}
&= 2k^3(k+1) + 6k^2(k+1) + 5k(k+1) + 1(k+1) \\
&= (k+1)[2k^3 + 6k^2 + 5k + 1] \\
&= (k+1)[2k^3 + 2k^2 + 4k^2 + 4k + k + 1] \\
&= (k+1)[2k^3(k+1) + 4k(k+1) + (k+1)] \\
&= (k+1)(k+1)(2k^2 + 4k + 1) \\
&= (k+1)^2(2k^2 + 4k + 2 - 1) \\
&= (k+1)^2[2(k^2 + 2k + 1) - 1] \\
&= (k+1)^2[2(k+1)^2 - 1]
\end{aligned}$$

Thus  $S(k+1)$  is true when  $S(k)$  is true.

Therefore condition 2 is satisfied.

Hence  $S(n)$  is true for every positive integer.

**Q.25 Use mathematical induction to prove that the following formula for every positive integer  $n$ .**

**$(x+1)$  is a factor of  $x^{2n} - 1$**

**Solution:**

Let  $S(n)$  be the given statement, that is

$S(n) = x+1$  is a factor of  $x^{2n} - 1$

**Condition 1**

When  $n = 1$ ,  $S(n)$  becomes

$$S(1) = x^2 - 1 = (x+1)(x-1)$$

Clearly  $x+1$  is a factor of  $x^2 - 1$ .

Therefore condition 1 is satisfied.

**Condition 2**

Let us suppose that  $S(n)$  is true for  $n = k \in \mathbb{N}$ , i.e.

$S(k)$ ;  $x+1$  is a factor of  $x^{2k} - 1$  ..... (1)

We want to prove that  $S(k+1)$  has a factor  $x+1$

$$S(k+1); x^{2(k+1)} - 1 = x^{2k+2} - 1 = x^{2k} x^2 - 1$$

Adding & subtracting  $x^2 \cdot 1$

$$\begin{aligned}
&= x^{2k} x^2 - x^2 - 1 + x^2 \cdot 1 - 1 \cdot 1 \\
&= x^2(x^{2k} - 1) + 1(x^2 - 1) \\
&= x^2(x^{2k} - 1) + 1(x-1)(x+1)
\end{aligned}$$

The first term has a factor  $x+1$  by assumption 1 & 2<sup>nd</sup> term clearly has a factor  $x+1$ . Thus the whole term has a factor  $x+1$ . Thus  $S(k+1)$  has a factor  $x+1$ . Therefore condition 2 is satisfied. Thus  $S(n)$  is true for all +ve integral values of  $n$ .



**Q.26 Use mathematical induction to prove that the following formula for every positive integer  $n$ .**

**$(x - y)$  is a factor of  $x^n - y^n$**

**Solution:**

Let  $S(n)$  be the given statement, that is

$S(n) = x - y$  is a factor of  $x^n - y^n$

**Condition 1**

When  $n = 1$ ,  $S(n)$  becomes

$S(1) = x - y$  is a factor of  $x^1 - y^1$ .

Therefore condition 1 is satisfied.

**Condition 2**

Let us suppose that  $S(n)$  is true for  $n = k \in \mathbb{N}$  i.e.

$S(k)$ ;  $x - y$  is a factor of  $x^k - y^k$  ..... (1)

We want to prove that  $S(k + 1)$  has a factor  $x - y$ .

$S(k + 1)$ ;  $x^{k+1} - y^{k+1}$

$x^k x - y^k y$

Subtracting & adding  $x y^k = x^k x - x y^k + x y^k - y^k y = x(x^k - y^k) + y^k(x - y)$

The first term has a factor  $x - y$  by substitution (1).

The 2<sup>nd</sup> term clearly has a factor  $x - y$ . Thus  $S(k + 1)$  has a factor  $x - y$ . Therefore condition 2 is satisfied. Hence  $S(n)$  is true for all +ve integers.

**Q.27 Use mathematical induction to prove that the following formula for every positive integer  $n$ .**

**$x + y$  is a factor of  $x^{2n-1} + y^{2n-1}$ .**

**Solution:**

Let  $S(n)$  be the given statement, i.e.

$S(n)$ ;  $x + y$  is a factor of  $x^{2n-1} + y^{2n-1}$



**Condition 1**

When  $n = 1$ ,  $S(n)$  becomes

$$S(1) \quad x + y \text{ is a factor of } x^{2-1} + y^{2-1} = x + y \text{ (True)}$$

Thus condition 1 is satisfied.

Let us suppose that  $S(n)$  is true for  $n = k \in \mathbb{N}$  i.e.

$$S(k); \quad x + y \text{ is a factor of } x^{2k-1} + y^{2k-1} \quad \dots\dots\dots (1)$$

We want to prove that  $S(k+1)$  has a factor  $x + y$

$$\begin{aligned} S(k+1); \quad x^{2(k+1)-1} + y^{2(k+1)-1} &= x^{2k+2-1} + y^{2k+2-1} \\ &= x^{2k-1} x^2 + y^{2k-1} y^2 \end{aligned}$$

Adding  $x^2 y^{2k-1}$  & subtracting

$$\begin{aligned} &= x^{2k-1} x^2 + x^2 y^{2k-1} - x^2 y^{2k-1} + y^{2k-1} y^2 \\ &= x^2 (x^{2k-1} + y^{2k-1}) - y^{2k-1} (x^2 - y^2) \end{aligned}$$

The first term has a factor  $x + y$  by (1). The second term clearly has a factor  $x + y$ . Thus  $S(k+1)$  has a factor  $x + y$ . Therefore condition 2 is satisfied. Therefore  $S(n)$  is true for all +ve integer  $n$ .

**Q.28 Use mathematical induction to show that**

$$1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1 \text{ for all non-negative integers } n.$$

**Solution:**

Let  $S(n)$  be the statement, i.e.

$$S(n): 1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1$$

**Condition 1**

When  $n = 0$ ,  $S(n)$  becomes

$$S(0): 1 = 2^0 - 1 = 2 - 1 = 1$$

$\Rightarrow$  L.H.S. = R.H.S.

Therefore condition 1 is satisfied.

**Condition 2**

Let us suppose that  $S(n)$  is true for  $n = k \in \mathbb{N}$ , i.e.

$$S(k): 1 + 2 + 2^2 + \dots + 2^k = 2^{k+1} - 1 \quad \dots\dots\dots (1)$$

Adding  $2^{k+1}$  on both sides

$$\begin{aligned} 1 + 2 + 2^2 + \dots + 2^k + 2^{k+1} &= 2^{k+1} - 1 + 2^{k+1} \\ &= 2 \cdot 2^{k+1} - 1 \\ &= 2^{k+1+1} - 1 \end{aligned}$$

Therefore condition 2 is satisfied.

Thus  $S(k+1)$  is true when  $S(k)$  is true.

Hence  $S(n)$  is true for all non-negative integer  $n$ .



**Q.29** If  $A$  and  $B$  are two matrices and  $AB = BA$ , then show by mathematical induction that  $AB^n = B^nA$  for any positive integers.

**Solution:**

Let  $S(n)$  the given statement, i.e.

$S(n): A \cdot B^n = B^nA$  for any +ve integer.

**Condition 1**

When  $n = 1$ ;  $S(n)$  becomes

$S(1): AB = BA$

$AB = BA$  (Given)

L.H.S. = R.H.S.

Therefore condition 1 is satisfied.

**Condition 2**

Let  $S(n)$  be true for  $n = k \in \mathbb{N}$ , that is  $S(k): AB^k = B^kA$  ..... (1)

We want to prove that  $S(k+1)$  is also true for that post multiply by  $B$ , we have

$$AB^k B = B^k A B$$

$$AB^{k+1} = B^k BA \quad (\text{Given } AB = BA)$$

$$AB^{k+1} = B^{k+1}A$$

Thus  $S(k+1)$  is true, when  $S(k)$  is true. Therefore condition 2 is satisfied.

Hence  $S(n)$  is true for any +ve integer.

**Q.30** Prove that by principle of mathematical induction that  $n^2 - 1$  is divisible by 8 when  $n$  is an odd positive integer.

**Solution:**

Let  $S(n)$  be the given statement i.e.

$S(n); n^2 - 1$  is divisible by 8.

**Condition 1**

When  $n = 1$ , we have

$$S(1) = 1^2 - 1 = 1 - 1 = 0 \text{ that is divisible by 8.}$$

Therefore condition 1 is satisfied.

$$S(k); k^2 - 1 \text{ is divisible by 8.} \quad \dots\dots\dots (1)$$



We want to prove that  $S(k + 1)$  is also true.

$$S(k + 1) ; (k + 1)^2 - 1$$

$$\begin{aligned} S(k + 2) &= (k + 1 + 1)^2 - 1 = (k + 2)^2 - 1 = k^2 + 4k + 4 - 1 \\ &= k^2 + 4k + 4 - 1 = (k^2 - 1) + 4(k + 1) \quad \dots\dots\dots (2) \end{aligned}$$

Clearly  $(k^2 - 1)$  is divisible by 8 by (1). As is odd +ve integer, so  $k + 1$  is an even integer. Hence  $k + 1 = 2m$  (say) where  $m \in \mathbb{Z}^+$ . Therefore (2) become  $(k^2 - 1) + 4(2m)$ .

$$= (k^2 - 1) + 8m$$

Therefore now  $8m$  is also clearly divisible by 8. Thus  $S(k + 2)$  is true when  $S(k)$  is true. Therefore condition (2) is satisfied. Hence  $S(n)$  is true.

**Q.31 Use mathematical induction to prove that  $\ln x^n = n \ln x$ , for any integer  $n \geq 0$ , if  $x$  is +ve number.**

**Solution:**

Let  $S(n)$  be the given statement that is  $S(n) ; \ln x^n = n \ln x$

**Condition 1:**

When,  $n = 0$   $S(n)$  becomes

$$S(0) ; \ln x^0 = 0 \ln x$$

$$\ln 1 = 0 \cdot \ln x$$

$$0 = 0 \Rightarrow \text{L.H.S} = \text{R.H.S}$$

Therefore condition 1 is satisfied for  $n \geq 0$ .

**Condition 2:**

Let the given statement  $S(n)$  be true for  $n = k \in \mathbb{N}$  i.e.

$$S(k); \ln x^k = k \ln x \quad \dots\dots\dots (1)$$

Adding  $\ln x$  on both sides

$$\ln x^k + \ln x = k \ln x + \ln x$$

$$\ln x^k x = (k + 1) \ln x$$

$$\ln x^{k+1} = (k + 1) \ln x$$

Thus  $S(k + 1)$  is true when  $S(k)$  is true.

Therefore condition 2 is satisfied.

Since both conditions are satisfied. Therefore  $S(n)$  is true for any integer  $n \geq 0$ .



**Q.32 Use the principle of extended mathematical induction to prove that  $n! > 2^n - 1$  for integral values of  $n \geq 4$ .**

**Solution:**

Let  $S(n)$  be the given statement, that is

$S(n): n! > 2^n - 1$  for integral values of  $n \geq 4$ .

**Condition 1**

When  $n = 4$ ,  $S(n)$  becomes

$S(4): 4! > 2^4 - 1$

$24 > 16 - 1$

$24 > 15$  which is true.

Therefore condition 1 is satisfied for  $n \geq 4$ .

**Condition 2**

Let us suppose that  $S(n)$  is true for  $n = k$

$S(k): k! > 2^k - 1$  for integral values of  $k \geq 4$  ..... (1)

Now multiplying throughout by  $k + 1$

$(k + 1) k! > (k + 1) (2^k - 1)$

$(k + 1)! > (k - 1 + 2) (2^k - 1)$

$> (k - 1) (2^k - 1) + 2 (2^k - 1)$

$> (k - 1) (2^k - 1) + 2^{k+1} - 2$

$> 2^{k+1} - 1 + [(k - 1) (2^k - 1) - 1]$

$\Rightarrow (k + 1)! > 2^{k+1} - 1 \quad \because \text{As } [(k - 1) (2^k - 1) - 1] \text{ is +ve so ignore it.}$

Thus  $S(k + 1)$  is true, when  $S(k)$  is true.

Therefore condition 2 is satisfied. Hence  $S(n)$  is true for all the integral values of  $n \geq 4$ .

**Q.33  $n^2 > n + 3$ , for integral values of  $n \geq 3$ . (Gujranawala Board, 2003)**

**Solution:**

Let  $S(n)$  be the given statement, that is

$S(n): n^2 > n + 3$  for integral values of  $n \geq 3$ .

**Condition 1**

When  $n = 3$ ,  $S(n)$  becomes

$S(3): 3^2 > 3 + 3$

$9 > 6$  which is true. Therefore condition 1 is satisfied for  $n \geq 3$ .



**Condition 2**

Let us suppose that  $S(n)$  is true for  $n = k$

$$S(k); k^2 > k + 3 \text{ for integral values of } k \geq 3 \quad \dots\dots\dots (1)$$

Adding  $2k + 1$ , throughout

$$k^2 + 2k + 1 > k + 3 + 2k + 1$$

$$(k + 1)^2 > k + 1 + 3 + (2k)$$

$$(\overline{k+1})^2 > \overline{k+1} + 3 \quad (\because \text{As } 2k \text{ is +ve, so ignore it})$$

Thus  $S(k + 1)$  is true, when  $S(k)$  is true.

Therefore condition 2 is satisfied. Therefore  $S(n)$  is true for  $n \geq 3$ .

**Q.34**  $4^n > 3^n + 2^{n-1}$ , for integral values of  $n \geq 2$ . (Lahore Board, 2003)

**Solution:**

Let  $S(n)$  be the given statement, that is

$$S(n): 4^n > 3^n + 2^{n-1}, \text{ for integral values of } n \geq 2.$$

**Condition 1**

When  $n = 2$  then  $S(n)$  becomes

$$S(2): 4^2 > 3^2 + 2^{2-1}$$

$$16 > 9 + 2$$

$$16 > 11 \text{ which is true, therefore condition 1 is true for } n \geq 2.$$

**Condition 2**

Let us suppose that  $S(n)$  is true for  $n = k$

$$S(k); 4^k > 3^k + 2^{k-1} \text{ for integral values of } k \geq 2 \quad \dots\dots\dots (1)$$

Multiplying throughout by 4

$$44^k > 4.3^k + 4.2^{k-1}$$

$$4^{k+1} > (3 + 1) 3^k + (2 + 2) 2^{k-1}$$

$$> 3^{k+1} + 3^k + 2^{k+1-1} + 2^{k-1+1}$$

$$> 3^{k+1} + 2^{k+1-1} + (3^k + 2^{k-1+1})$$

$$\overline{4^{k+1}} > \overline{3^{k+1}} + \overline{2^{k+1-1}} \quad (\because \text{As } 3^k + 2^k \text{ is +ve so ignore it})$$

Thus  $S(k + 1)$  is true, when  $S(k)$  is true. Therefore condition 2 is satisfied.

Hence  $S(n)$  is true for the integral values of  $n \geq 2$ .



**Q.35  $3^n < n!$  for integral values of  $n > 6$ .****Solution:**

Let  $S(n)$  be the given statement, that is

$S(n): 3^n < n!$  for integral values of  $n > 6$

**Condition 1**

When  $n = 7$ ,  $S(n)$  becomes

$S(7); 3^7 < 7!$

$2187 < 5040$  which is true. Condition is satisfied.

**Condition 2**

Let us suppose that  $S(n)$  is true for  $n = k$

$S(k); 3^k < k!$  for integral values of  $k > 6$  ..... (1)

Multiplying throughout by  $(k + 1)$

$(k + 1) 3^k < (k + 1) k!$

$(k - 2 + 3) 3^k < (k + 1)!$

$(k - 2) 3^k + 3^{k+1} < (k + 1)!$

$3^{k+1} < (k + 1)!$  ( $\because$  As  $(k - 2) 3^k$  is +ve, so ignore it)

Thus  $S(k + 1)$  is true, when  $S(k)$  is true. Therefore condition 2 is satisfied.

Since both conditions are satisfied.

Therefore,  $S(n)$  is true for integral values of  $n > 6$ .

**Q.36  $n! > n^2$  for integral values of  $n \geq 4$ .****Solution:**

Let  $S(n)$  be the given statement, that is

$S(n); n! > n^2$  for integral values of  $n \geq 4$

**Condition 1**

When  $n = 4$ ,  $S(n)$  becomes

$S(4); 4! > 4^2$

$24 > 16$  which is true. Therefore condition 1 is satisfied for  $n \geq 4$ .

**Condition 2**

Let us suppose that  $S(n)$  is true for  $n = k$ .

$S(k); k! > k^2$  for integral values of  $k \geq 4$  ..... (1)



Multiplying throughout by  $k + 1$

$$(k + 1) k! > (k + 1) k^2$$

$$(k + 1)! > k^3 + k^2$$

$$(k + 1)! > k^2 + 2k + 1 + (k^3 - 2k - 1)$$

$$(k + 1)! > (k + 1)^2 \quad (\because \text{As } k^3 - 2k - 1 \text{ is +ve so ignore it})$$

Thus  $S(k + 1)$  is true, when  $S(k)$  is true.

Therefore condition 2 is satisfied. Hence  $S(n)$  is true for integral values of  $n \geq 4$ .

**Q.37**  $3 + 5 + 7 + \dots + (2n + 5) = (n + 2)(n + 4)$  for integral values of  $n \geq -1$ .

**Solution:**

Let  $S(n)$  be the given statement, that is

$$S(n): 3 + 5 + 7 + \dots + (2n + 5) = (n + 2)(n + 4), \text{ for integral values of } n \geq -1$$

**Condition 1**

When  $n = -1$ ,  $S(n)$  becomes

$$S(-1); 3 = (-1 + 2)(-1 + 4)$$

$$3 = 3$$

$\Rightarrow$  L.H.S. = R.H.S. Therefore condition 1 is satisfied for  $n \geq -1$ .

**Condition 2**

Let us suppose that statement is true for  $n = k$ .

$$3 + 5 + 7 + \dots + (2k + 5) = (k + 2)(k + 4) \quad \dots \dots \dots (1)$$

Adding  $(2(k + 1) + 5)$  on both sides

$$\begin{aligned} 3 + 5 + 7 + \dots + (2k + 5) + (2k + 7) &= (k + 2)(k + 4) + (2k + 7) \\ &= k^2 + 4k + 2k + 8 + 2k + 7 \\ &= k^2 + 8k + 15 \\ &= k^2 + 5k + 3k + 15 \\ &= (k + 5)(k + 3) \\ &= (k + 1 + 4)(k + 1 + 2) \end{aligned}$$

Condition 2 is satisfied. Thus  $S(k + 1)$  is true when  $S(k)$  is true.

Hence  $S(n)$  is true for integral values of  $n \geq -1$ .



**Q.38**  $1 + nx \leq (1 + x)^n$  for  $n \geq 2$  &  $x > -1$

**Solution:**

Let  $S(n)$  be the given statement, i.e.

$S(n): 1 + nx \leq (1 + x)^n$  for  $n \geq 2$  &  $x > -1$

**Condition 1**

When  $n = 2$ ,  $S(n)$  becomes

$S(2); 1 + 2x \leq (1 + x)^2$

$1 + 2x \leq 1 + x^2 + 2x$ , which is true.

Therefore condition 1 is satisfied for  $n \geq 2$  and  $x > -1$ .

**Condition 2**

Let us suppose that  $S(n)$  is true for  $n = k$

$S(k); 1 + kx \leq (1 + x)^k$  for  $k \geq 2$  and  $x > -1$  ..... (1)

Multiplying throughout by  $(1 + x)$

$(1 + kx)(1 + x) \leq (1 + x)^k (1 + x)$

$1 + x + kx + kx^2 \leq (1 + x)^{k+1}$

$1 + (k + 1)x + kx^2 \leq (1 + x)^{k+1}$

$1 + (k + 1)x \leq (1 + x)^{k+1}$  ( $\because kx^2$  is +ve, so ignore it)

Thus  $S(k + 1)$  is true when  $S(k)$  is true.

Therefore condition 2 is satisfied.

Therefore  $S(n)$  is true for all  $n \geq 2$  and  $x > -1$ .

**BINOMIAL THEOREM (Definition Lahore Board 2010)**

An algebraic expression consisting of two terms such as  $a + x$ ,  $x - 2y$ ,  $ax + b$  etc. is called a binomial or binomial expression.

We know that  $(a + x)^2 = a^2 + 2ax + x^2$

The R.H.S. is called **binomial expansion** and 2 is called index.

**FORMULA OF BINOMIAL THEOREM (Lahore Board 2009–11)**

$$(a + x)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \dots + \binom{n}{r-1}a^{n-r+1}x^{r-1} \\ + \binom{n}{r}a^{n-r}x^r + \dots + \binom{n}{n}x^n$$

it can briefly written as

$$(a + x)^n = \sum_{r=0}^n \binom{n}{r}a^{n-r}x^r$$



**NOTE:**

- (i) The rule or formula of a binomial for expansion raised to any positive integral power  $n$ .
- (ii) It is finite series.
- (iii) Number of terms in the expansion of  $(a + x)^n$  is  $n + 1$ .
- (iv)  $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n}$  are called binomial coefficients.
- (v)  $\binom{n}{0}, \binom{n}{2}, \binom{n}{4}, \dots, \binom{n}{n}$  are called even binomial coefficients.
- (vi)  $\binom{n}{1}, \binom{n}{3}, \binom{n}{5}, \dots, \binom{n}{n-1}$  are called odd binomial coefficients.

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**SUM OF BINOMIAL COEFFICIENTS**

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n$$

**SUM OF EVEN BINOMIAL COEFFICIENTS**

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots + \binom{n}{n} = 2^{n-1}$$

**SUM OF ODD BINOMIAL COEFFICIENTS**

$$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots + \binom{n}{n-1} = 2^{n-1}$$

**REMARK**

Sum of even binomial coefficients = Sum of odd binomial coefficients.

**EXERCISE 8.2**

**Q.1** Using binomial theorem, expand the following:

(i)  $(a + 2b)^5$

(ii)  $\left(\frac{x}{2} - \frac{2}{x^2}\right)^6$

(iii)  $\left(3a - \frac{x}{3a}\right)^4$

(iv)  $\left(2a - \frac{x}{a}\right)^7$

(v)  $\left(\frac{x}{2y} - \frac{2y}{x}\right)^8$

(vi)  $\left[\sqrt{\frac{a}{x}} - \sqrt{\frac{x}{a}}\right]^6$

**Solution:**

(i)  $(a + 2b)^5$

$$\begin{aligned} &= \binom{5}{0}a^5(2b)^0 + \binom{5}{1}a^4(2b)^1 + \binom{5}{2}a^3(2b)^2 + \binom{5}{3}a^2(2b)^3 + \binom{5}{4}a^1(2b)^4 + \binom{5}{5}a^0(2b)^5 \\ &= a^5 + 5a^4(2b) + 10a^3(4b^2) + 10a^2(8b^3) + 5a(16b^4) + 32b^5 \\ &= a^5 + 10a^4b + 40a^3b^2 + 80a^2b^3 + 80ab^4 + 32b^5 \end{aligned}$$



$$\begin{aligned}
 \text{(ii)} \quad & \left( \frac{x}{2} - \frac{2}{x^2} \right)^6 \\
 &= \binom{6}{0} \left( \frac{x}{2} \right)^6 - \binom{6}{1} \left( \frac{x}{2} \right)^{6-1} \left( \frac{2}{x^2} \right)^1 + \binom{6}{2} \left( \frac{x}{2} \right)^{6-2} \left( \frac{2}{x^2} \right)^2 - \binom{6}{3} \left( \frac{x}{2} \right)^{6-3} \left( \frac{2}{x^2} \right)^3 \\
 &\quad + \binom{6}{4} \left( \frac{x}{2} \right)^{6-4} \left( \frac{2}{x^2} \right)^4 - \binom{6}{5} \left( \frac{x}{2} \right)^{6-5} \left( \frac{2}{x^2} \right)^5 + \binom{6}{6} \left( \frac{x}{2} \right)^{6-6} \left( \frac{2}{x^2} \right)^6 \\
 &= \frac{x^6}{64} - 6 \frac{x^5}{32} \times \frac{2}{x^2} + 15 \times \frac{x^4}{16} \times \frac{4}{x^4} - 20 \times \frac{x^3}{8} \times \frac{8}{x^6} + 15 \times \frac{x^2}{4} \times \frac{16}{x^8} - 6 \times \frac{x}{2} \times \frac{32}{x^{10}} + \frac{64}{x^{12}} \\
 &= \frac{x^6}{64} - \frac{3}{8} x^3 + \frac{15}{4} - \frac{20}{x^3} + \frac{60}{x^6} - \frac{96}{x^9} + \frac{64}{x^{12}}
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad & \left( 3a - \frac{x}{3a} \right)^4 \\
 &= \binom{4}{0} (3a)^4 \left( \frac{x}{3a} \right)^0 - \binom{4}{1} (3a)^{4-1} \left( \frac{x}{3a} \right)^1 + \binom{4}{2} (3a)^{4-2} \left( \frac{x}{3a} \right)^2 - \binom{4}{3} (3a)^{4-3} \left( \frac{x}{3a} \right)^3 \\
 &\quad + \binom{4}{4} (3a)^{4-4} \left( \frac{x}{3a} \right)^4 \\
 &= 81a^4 - 4 \times 27a^3 \times \frac{x}{3a} + 6 \times 9a^2 \times \frac{x^2}{9a^2} \times \frac{x^2}{9a^2} - 4 \times 3a \times \frac{x^3}{27a^3} + \frac{x^4}{81a^4} \\
 &= 81a^4 - \frac{108}{3} a^2 x + 6x^2 - \frac{4x^3}{9a^2} + \frac{x^4}{81a^4} \\
 &= 81a^4 - 36a^2 x + 6x^2 - \frac{4x^3}{9a^2} + \frac{x^4}{81a^4}
 \end{aligned}$$

$$\begin{aligned}
 \text{(iv)} \quad & \left( 2a - \frac{x}{a} \right)^7 \\
 &= \binom{7}{0} (2a)^7 - \binom{7}{1} (2a)^6 \left( \frac{x}{a} \right)^1 + \binom{7}{2} (2a)^5 \left( \frac{x}{a} \right)^2 - \binom{7}{3} (2a)^4 \left( \frac{x}{a} \right)^3 \\
 &\quad + \binom{7}{4} (2a)^3 \left( \frac{x}{a} \right)^4 - \binom{7}{5} (2a)^2 \left( \frac{x}{a} \right)^5 + \binom{7}{6} (2a)^1 \left( \frac{x}{a} \right)^6 - \binom{7}{7} (2a)^0 \left( \frac{x}{a} \right)^7 \\
 &= 128a^7 - 7(64a^6) \left( \frac{x}{a} \right) + 21 \times 32a^5 \frac{x^2}{a^2} - 35(16a^4) \left( \frac{x^3}{a^3} \right) \\
 &\quad + 35(8a^3) \left( \frac{x^4}{a^4} \right) - 21(4a^2) \left( \frac{x^5}{a^5} \right) + 7(2a) \left( \frac{x^6}{a^6} \right) - \frac{x^7}{a^7} \\
 &= 128a^7 - 448x a^5 + 672a^3 x^2 - 560a x^3 + \frac{280x^4}{a} - \frac{84x^5}{a^3} + \frac{14x^6}{a^5} - \frac{x^7}{a^7}
 \end{aligned}$$



$$\begin{aligned}
 \text{(v)} \quad & \left( \frac{x}{2y} - \frac{2y}{x} \right)^8 \\
 &= \binom{8}{0} \left( \frac{x}{2y} \right)^8 - \binom{8}{1} \left( \frac{x}{2y} \right)^7 \left( + \frac{2y}{x} \right) + \binom{8}{2} \left( \frac{x}{2y} \right)^6 \left( \frac{2y}{x} \right)^2 - \binom{8}{3} \left( \frac{x}{2y} \right)^5 \left( \frac{2y}{x} \right)^3 \\
 &\quad + \binom{8}{4} \left( \frac{x}{2y} \right)^4 \left( \frac{2y}{x} \right)^4 - \binom{8}{5} \left( \frac{x}{2y} \right)^3 \left( \frac{2y}{x} \right)^5 + \binom{8}{6} \left( \frac{x}{2y} \right)^2 \left( \frac{2y}{x} \right)^6 \\
 &\quad - \binom{8}{7} \left( \frac{x}{2y} \right)^1 \left( \frac{2y}{x} \right)^7 + \binom{8}{8} \left( \frac{x}{2y} \right)^0 \left( \frac{2y}{x} \right)^8 \\
 &= \frac{x^8}{256y^8} - 8 \frac{x^7}{128y^7} \times \frac{2y}{x} + 28 \left( \frac{x^6}{64y^6} \right) \left( \frac{4y^2}{x^2} \right) - 56 \left( \frac{x^5}{32y^5} \right) \left( \frac{8y^3}{x^3} \right) \\
 &\quad + 70 \left( \frac{x^4}{16y^4} \right) \left( \frac{16y^4}{x^4} \right) - 56 \left( \frac{x^3}{8y^3} \right) \left( \frac{32y^5}{x^5} \right) + 28 \frac{x^2}{4y^2} \times \frac{64y^6}{x^6} - 8 \times \frac{x}{2y} \times \frac{128y^7}{x^7} + \frac{256y^8}{x^8} \\
 &= \frac{x^8}{256y^8} - \frac{x^6}{8y^6} + \frac{7x^4}{16y^4} - 14 \frac{x^2}{y^2} + 70 - \frac{224y^2}{x^2} + \frac{448y^4}{x^4} - \frac{512y^6}{x^6} + \frac{256y^8}{x^8}
 \end{aligned}$$

$$\begin{aligned}
 \text{(vi)} \quad & \left[ \sqrt{\frac{a}{x}} - \sqrt{\frac{x}{a}} \right]^6 \\
 &= \binom{6}{0} \left( \sqrt{\frac{a}{x}} \right)^6 - \binom{6}{1} \left( \sqrt{\frac{a}{x}} \right)^{6-1} \left( \sqrt{\frac{x}{a}} \right)^1 \\
 &\quad + \binom{6}{2} \left( \sqrt{\frac{a}{x}} \right)^{6-2} \left( \sqrt{\frac{x}{a}} \right)^2 - \binom{6}{3} \left( \sqrt{\frac{a}{x}} \right)^{6-3} \left( \sqrt{\frac{x}{a}} \right)^3 \\
 &\quad + \binom{6}{4} \left( \sqrt{\frac{a}{x}} \right)^{6-4} \left( \sqrt{\frac{x}{a}} \right)^4 - \binom{6}{5} \left( \sqrt{\frac{a}{x}} \right)^{6-5} \left( \sqrt{\frac{x}{a}} \right)^5 + \binom{6}{6} \left( \sqrt{\frac{a}{x}} \right)^0 \left( \sqrt{\frac{x}{a}} \right)^6 \\
 &= \left( \sqrt{\frac{a}{x}} \right)^6 - 6 \left( \sqrt{\frac{a}{x}} \right)^5 \left( \sqrt{\frac{x}{a}} \right)^1 + 15 \left( \sqrt{\frac{a}{x}} \right)^4 \left( \sqrt{\frac{x}{a}} \right)^2 \\
 &\quad - 20 \left( \sqrt{\frac{a}{x}} \right)^3 \left( \sqrt{\frac{x}{a}} \right)^3 + 15 \left( \sqrt{\frac{a}{x}} \right)^2 \left( \sqrt{\frac{x}{a}} \right)^4 - 6 \left( \sqrt{\frac{a}{x}} \right) \left( \sqrt{\frac{x}{a}} \right)^5 + \left( \sqrt{\frac{x}{a}} \right)^6 \\
 &= \left( \sqrt{\frac{a}{x}} \right)^6 - 6 \left( \sqrt{\frac{a}{x}} \right)^4 + 15 \left( \sqrt{\frac{a}{x}} \right)^2 - 20 \left( \sqrt{\frac{a}{x}} \right)^0 \\
 &\quad + 15 \left( \sqrt{\frac{a}{x}} \right)^{-2} - 6 \left( \sqrt{\frac{a}{x}} \right)^{-4} + \left( \sqrt{\frac{x}{a}} \right)^6 \\
 &= \left( \frac{a}{x} \right)^{6/2} - 6 \left( \frac{a}{x} \right)^{4/2} + 15 \left( \frac{a}{x} \right) - 20 + 15 \left( \frac{a}{x} \right)^{-2/2} - 6 \left( \frac{a}{x} \right)^{-4/2} + \left( \frac{x}{a} \right)^{6/2} \\
 &= \frac{a^3}{x^3} - 6 \frac{a^2}{x^2} + 15 \frac{a}{x} - 20 + 15 \left( \frac{a}{x} \right)^{-1} - 6 \left( \frac{a}{x} \right)^{-2} + \left( \frac{x}{a} \right)^3 \\
 &= \frac{a^3}{x^3} - 6 \frac{a^2}{x^2} + 15 \frac{a}{x} - 20 + 15 \frac{x}{a} - 6 \frac{x^2}{a^2} + \frac{x^3}{a^3}
 \end{aligned}$$



**Q.2 Calculate the following by means of binomial theorem.**

(i)  $(0.97)^3$  (Lahore Board 2010)

(ii)  $(2.02)^4$  (Lahore Board 2011)

(iii)  $(9.98)^4$

(iv)  $(2.9)^5$

**Solution:**

(i)  $(0.97)^3 = (1 - 0.03)^3$

$$= \binom{3}{0}(1)^3 - \binom{3}{1}(1)^2(0.03)^1 + \binom{3}{2}(1)^1(0.03)^2 - \binom{3}{3}(1)^0(0.03)^3$$

$$= 1 - 0.09 + 0.0027 - 0.000027$$

$$= 0.9127$$

(ii)  $(2.02)^4 = (2 + 0.02)^4$

$$= \binom{4}{0}(2)^4 + \binom{4}{1}(2)^3(0.02)^1 + \binom{4}{2}(2)^2(0.02)^2 + \binom{4}{3}(2)^1(0.02)^3 + \binom{4}{4}(2)^0(0.02)^4$$

$$= 16 + 4(8)(0.02) + 6(4)(0.0004) + 4(2)(0.000008) + 0.00000016$$

$$= 16.64 + 0.0096 + 0.000064$$

$$= 16.64$$

(iii)  $(9.98)^4 = (10 - 0.02)^4$

$$= \binom{4}{0}(10)^4(0.02)^0 - \binom{4}{1}(10)^3(0.02)^1 + \binom{4}{2}(10)^2(0.02)^2 - \binom{4}{3}(10)^1(0.02)^3 + \binom{4}{4}(10)^0(0.02)^4$$

$$= 10000 - 80 + 600(0.0004) - 40(0.000008) + 0.00000016$$

$$= 9920.24$$

(iv)  $(2.9)^5 = (3 - 0.1)^5$

$$= \binom{5}{0}(3)^5 - \binom{5}{1}(3)^4(0.1) + \binom{5}{2}(3)^3(0.1)^2 - \binom{5}{3}(3)^2(0.1)^3 + \binom{5}{4}(3)^1(0.1)^4 - \binom{5}{5}(3)^0(0.1)^5$$

$$= 243 - 4.05 + 10(27)(0.0001) - 10(9)(0.000001) + 15(0.00000001) - 0.0000000001$$

$$= 24.3 + 5 \times 81 - 0.01 + 10 \times 8 \times 0.0001$$

$$= 205.2$$



**Q.3 Expand and simplify the following:**

- (i)  $(a + \sqrt{2}x)^4 + (a - \sqrt{2}x)^4$   
 (ii)  $(2 + \sqrt{3})^5 + (2 - \sqrt{3})^5$   
 (iii)  $(2 + i)^5 - (2 - i)^5$   
 (iv)  $(x + \sqrt{x^2 - 1})^3 + (x - \sqrt{x^2 - 1})^3$

**Solution:**

(i)  $(a + \sqrt{2}x)^4 + (a - \sqrt{2}x)^4$   
 $= \binom{4}{0}a^4 + \binom{4}{1}a^3(\sqrt{2}x)^1 + \binom{4}{2}a^2(\sqrt{2}x)^2 + \binom{4}{3}a(\sqrt{2}x)^3 + \binom{4}{4}a^0(\sqrt{2}x)^4$   
 $(a + \sqrt{2}x)^4 = a^4 + 4a^3\sqrt{2}x + 6a^2(\sqrt{2}x)^2 + 4a(\sqrt{2}x)^3 + (\sqrt{2}x)^4 \dots\dots\dots (i)$   
 $(a - \sqrt{2}x)^4 = a^4 - 4a^3\sqrt{2}x + 6a^2(\sqrt{2}x)^2 - 4a(\sqrt{2}x)^3 + (\sqrt{2}x)^4 \dots\dots\dots (ii)$

By adding (i) and (ii)

$$\begin{aligned} (a + \sqrt{2}x)^4 + (a - \sqrt{2}x)^4 &= 2a^4 + 12a^2(\sqrt{2}x)^2 + 2(\sqrt{2}x)^4 \\ &= 2a^4 + 12a^2(2x^2) + 2(4x^4) \\ &= 2a^4 + 24a^2x^2 + 8x^4 \end{aligned}$$

(ii)  $(2 + \sqrt{3})^5 + (2 - \sqrt{3})^5$   
 $(2 + \sqrt{3})^5 = \binom{5}{0}(2)^5 + \binom{5}{1}(2)^4(\sqrt{3})^1 + \binom{5}{2}(2)^3(\sqrt{3})^2 + \binom{5}{3}(2)^2(\sqrt{3})^3$   
 $+ \binom{5}{4}(2)(\sqrt{3})^4 + \binom{5}{5}(2)^0(\sqrt{3})^5$   
 $= 32 + 5 \times 16\sqrt{3} + 10 \times 8(\sqrt{3})^2 + 10 \times 4(\sqrt{3})^3 + 5(2)(\sqrt{3})^4 + (\sqrt{3})^5$   
 $= 32 + 80\sqrt{3} + 80(\sqrt{3})^2 + 40(\sqrt{3})^3 + 10(\sqrt{3})^4 + (\sqrt{3})^5$   
 $(2 - \sqrt{3})^5 = 32 - 80\sqrt{3} + 80(\sqrt{3})^2 - 40(\sqrt{3})^3 + 10(\sqrt{3})^4 - (\sqrt{3})^5$

Adding

$$(2 + \sqrt{3})^5 + (2 - \sqrt{3})^5 = 64 + 480 + 180 = 724$$

(iii)  $(2 + i)^5 - (2 - i)^5$   
 $(2 + i)^5 = \binom{5}{0}(2)^5 + \binom{5}{1}(2)^4(i) + \binom{5}{2}(2)^3(i)^2 + \binom{5}{3}(2)^2(i)^3$   
 $+ \binom{5}{4}(2)^1(i)^4 + \binom{5}{5}(2)^0(i)^5$



$$\begin{aligned}
 &= 32 + 5 \times 16 (i) + 10 \times 8 \times (i)^2 + 10 \times 4 (i)^3 + 5 \times 2 (i)^4 + i^5 \\
 &= 32 + 80i + 80i^2 + 40i^3 + 10i^4 + i^5 \\
 (2-i)^5 &= 32 - 80i + 80i^2 - 40i^3 + 10i^4 - i^5 \\
 &\quad - \quad + \quad - \quad + \quad - \quad +
 \end{aligned}$$

Subtracting

$$\begin{aligned}
 (2+i)^5 - (2-i)^5 &= 160i + 80i^3 + 2i^5 \\
 &= 160i - 80i + 2i = 82i
 \end{aligned}$$

(iv)  $(x + \sqrt{x^2 - 1})^3 + (x - \sqrt{x^2 - 1})^3$

First we take  $(x + \sqrt{x^2 - 1})^3$

$$= x^3 + \binom{3}{1}(x)^{3-1}(\sqrt{x^2 - 1}) + \binom{3}{2}(x)^1(\sqrt{x^2 - 1})^2 + \binom{3}{3}(x)^0(\sqrt{x^2 - 1})^3$$

$$(x + \sqrt{x^2 - 1})^3 = x^3 + 3x^2\sqrt{x^2 - 1} + 3x(x^2 - 1) + (\sqrt{x^2 - 1})^3$$

$$(x - \sqrt{x^2 - 1})^3 = x^3 - 3x^2\sqrt{x^2 - 1} + 3x(x^2 - 1) - (\sqrt{x^2 - 1})^3$$

Adding

$$\begin{aligned}
 (x + \sqrt{x^2 - 1})^3 + (x - \sqrt{x^2 - 1})^3 &= 2x^3 + 6x(x^2 - 1) \\
 &= 2x^3 + 6x^3 - 6x \\
 &= 8x^3 - 6x
 \end{aligned}$$

**Q.4 Expand the following in ascending power of x**

(i)  $(2 + x - x^2)^4$

(ii)  $(1 - x + x^2)^4$

(iii)  $(1 - x - x^2)^4$

**Solution:**

(i)  $(2 + x - x^2)^4$

Let  $2 + x = y$

$$\begin{aligned}
 (y - x^2)^4 &= \binom{4}{0}(y)^4(x^2)^0 - \binom{4}{1}(y^3)(x^2)^1 + \binom{4}{2}(y^2)(x^2)^2 - \binom{4}{3}(y)(x^2)^3 + \binom{4}{4}(y)^0(x^2)^4 \\
 &= y^4 - 4y^3x^2 + 6y^2x^4 - 4yx^6 + x^8
 \end{aligned}$$

Putting value  $y = 2 + x$  again

$$= (2 + x)^4 - 4(2 + x)^3x^2 + 6(2 + x)^2x^4 - 4(2 + x)x^6 + x^8$$

$$= \left[ \binom{4}{0}(2)^4 + \binom{4}{1}(2)^3(x) + \binom{4}{2}(2)^2(x)^2 + \binom{4}{3}(2)^1(x)^3 + \binom{4}{4}(2)^0(x)^4 \right]$$

$$- 4[8 + x^3 + 6x^2 + 12x]x^2 + 6(4 + x^2 + 4x)x^4 - (8 + 4x)x^6 + x^8$$

$$= 16 + 32x - 8x^2 - 40x^3 + x^4 + 20x^5 - 2x^6 - 4x^7 + x^8$$



(ii)  $(1 - x + x^2)^4$

Let  $1 - x = y$

$$\begin{aligned}(y + x^2)^4 &= \binom{4}{0}y^4 + \binom{4}{1}y^3 x^2 + \binom{4}{2}y^2 (x^2)^2 + \binom{4}{3}y (x^2)^3 + \binom{4}{4}y^0 (x^2)^4 \\&= y^4 + 4y^3 x^2 + 6y^2 x^4 + 4y x^6 + x^8\end{aligned}$$

Putting value of  $y$

$$\begin{aligned}&= (1 - x)^4 + 4(1 - x)^3 x^2 + 6(1 - x)^2 x^4 + 4(1 - x) x^6 + x^8 \\&= \left[ \binom{4}{0}(1)^4 (x)^0 - \binom{4}{1}(1)^3 (x)^1 + \binom{4}{2}(1)^2 (x)^2 - \binom{4}{3}(1)^1 (x)^3 + \binom{4}{4}(1)^0 (x)^4 \right] \\&\quad + 4[1 - x^3 - 3x + 3x^2] x^2 + 6(1 + x^2 - 2x) x^4 + 4(x^6 - x^7) + x^8 \\&= 1 - 4x + 6x^2 - 4x^3 + x^4 + 4x^2 - 12x^3 + 12x^4 - 4x^5 + 6x^4 - 12x^5 + 10x^6 - 4x^7 + x^8 \\&= 1 - 4x + 10x^2 - 16x^3 + 19x^4 - 4x^5 + 10x^6 - 4x^7 + x^8\end{aligned}$$

(iii)  $(1 - x - x^2)^4$

Let  $1 - x = y$

$$\begin{aligned}(y - x^2)^4 &= \binom{4}{0}y^4 - \binom{4}{1}y^3 x^2 + \binom{4}{2}y^2 (x^2)^2 - \binom{4}{3}y (x^2)^3 + \binom{4}{4}y^0 (x^2)^4 \\&= y^4 - 4y^3 x^2 + 6y^2 x^4 - 4x^6 y + x^8\end{aligned}$$

Putting value of  $y$

$$\begin{aligned}&= (1 - x)^4 - 4(1 - x)^3 x^2 + 6(1 - x)^2 x^4 - 4x^6 (1 - x) + x^8 \\&= \left[ \binom{4}{0}(1)^4 - \binom{4}{1}(1)^3 (x) + \binom{4}{2}(1)^2 (x)^2 - \binom{4}{3}(1)(x)^3 + \binom{4}{4}(1)^0 (x)^4 \right] \\&\quad - 4[1 - x^3 - 3x + 3x^2] x^2 + 6(1 + x^2 - 2x) x^4 - 4(x^6 - x^7) + x^8 \\&= 1 - 4x + 6x^2 - 4x^3 + x^4 - 4x^2(1 - 3x + 3x^2 - x^3) + 6x^4 - 12x^5 + 6x^6 - 4x^6 + 4x^7 + x^8 \\&= 1 - 4x + 2x^2 + 8x^3 - 5x^4 - 8x^5 + 2x^6 + 4x^7 + x^8\end{aligned}$$

**Q.5 Expand the following in descending power of  $x$**

(i)  $(x^2 + x - 1)^3$

(ii)  $\left(x - 1 - \frac{1}{x}\right)^3$

**Solution:**

(i)  $(x^2 + x - 1)^3$

Let  $x - 1 = y \Rightarrow (x^2 + y)^3$

$$= \binom{3}{0}(x^2)^3 + \binom{3}{1}(x^2)^2 (y) + \binom{3}{2}(x^2) (y)^2 + \binom{3}{3}(x^2)^0 (y)^3$$

$$(x^2 + y)^3 = x^6 + 3x^4 y + 3x^2 y^2 + y^3$$



Putting value of  $y$

$$\begin{aligned}(x^2 + x - 1)^3 &= x^6 + 3x^4(x-1) + 3x^2(x-1)^2 + (x-1)^3 \\&= x^6 + 3x^5 - 3x^4 + 3x^2(x^2 + 1 - 2x) + x^3 - 1 - 3x^2 + 3x \\&= x^6 + 3x^5 - 3x^4 + 3x^4 + 3x^2 - 6x^3 + x^3 - 1 - 3x^2 + 3x \\&= x^6 + 3x^5 - 5x^3 + 3x - 1\end{aligned}$$

(i)  $\left(x - 1 - \frac{1}{x}\right)^3$

$$\begin{aligned}\text{Let } x - 1 &= y \Rightarrow \left(y - \frac{1}{x}\right)^3 \\&= \binom{3}{0}(y^3) - \binom{3}{1}(y^2)\left(\frac{1}{x}\right) + \binom{3}{2}(y)\left(\frac{1}{x}\right)^2 - \binom{3}{3}(y)^0\left(\frac{1}{x}\right)^3 \\&= y^3 - \frac{3y^2}{x} + \frac{3y}{x^2} - \frac{1}{x^3}\end{aligned}$$

Putting value of  $y$

$$\begin{aligned}\left(x - 1 - \frac{1}{x}\right)^3 &= (x-1)^3 - \frac{3(x-1)^2}{x} + \frac{3(x-1)}{x^2} - \frac{1}{x^3} \\&= x^3 - 1 - 3x^2 + 3x - \frac{3}{x}(x^2 + 1 - 2x) + \frac{1}{x^2}(3x - 3) - \frac{1}{x^3} \\&= x - 3x^2 + 5 - \frac{3}{x^2} - \frac{1}{x^3}\end{aligned}$$

### GENERAL TERM OF EXPANSION $(a + x)^n$

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

**Q.6 Find the term involving:**

- (i)  $x^4$  in the expansion of  $(3 - 2x)^7$
- (ii)  $x^{-2}$  in the expansion of  $\left(x - \frac{2}{x^2}\right)^{13}$
- (iii)  $a^4$  in the expansion of  $\left(\frac{2}{x} - a\right)^9$
- (iv)  $y^3$  in the expansion of  $(x - \sqrt{y})^{11}$

(Lahore Board 2008)

(Lahore Board 2004)

**Solution:**

(i)  $x^4$  in the expansion of  $(3 - 2x)^7$

We know that general term formula is

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$



Since  $n = 7$ ,  $a = 3$ ,  $x = (-2x)$

$$T_{r+1} = \binom{7}{r} a^{7-r} (-2x)^r$$

$$T_{r+1} = \binom{7}{r} 3^{7-r} (-2)^r x^r \quad \dots\dots\dots (1)$$

We have to find term involving  $x^4$ , so comparing the powers of  $x$ , we have  $r = 4$

Putting  $r = 4$  in (1)

$$T_{4+1} = \binom{7}{4} 3^{7-4} (-2)^4 x^4$$

$$= 35 \times 27 \times 16x^4$$

$$T_{4+1} = 15120 x^4$$

(ii)  $x^{-2}$  in the expansion of  $\left(x - \frac{2}{x^2}\right)^{13}$

We know that general term formula

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$a = x, n = 13, x = \left(-\frac{2}{x^2}\right)$$

$$T_{r+1} = \binom{13}{r} (x)^{13-r} \left(-\frac{2}{x^2}\right)^r$$

$$= \binom{13}{r} x^{13-r-2r} (-2)^r$$

$$= \binom{13}{r} x^{13-3r} (-2)^r \quad \dots\dots\dots (1)$$

We have to find term involving  $x^{-2}$  so comparing the power of  $x$  in (1)

$$13 - 3r = -2$$

$$13 + 2 = 3r$$

$$15 = 3r$$

$$\boxed{r = 5}$$

Put in (1)

$$T_{5+1} = \binom{13}{5} x^{13-3(5)} (-2)^5$$

$$T_6 = 1287 \times x^{-2} \times -32 = -41184 x^{-2}$$



**(iii)  $a^4$  in the expansion of  $\left(\frac{2}{x} - a\right)^9$** 

We know that general term formula

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$n = 9, a = \frac{2}{x}, x = (-a)$$

$$\begin{aligned} T_{r+1} &= \binom{9}{r} \left(\frac{2}{x}\right)^{9-r} (-a)^r \\ &= \binom{9}{r} \left(\frac{2}{x}\right)^{9-r} (-1)^r a^r \quad \dots\dots\dots (1) \end{aligned}$$

We have to find term involving  $a^4$ , so comparing the powers of  $a$ , we get

$$\begin{aligned} T_{4+1} &= \binom{9}{4} \left(\frac{2}{x}\right)^{9-4} (-1)^4 (a)^4 \\ &= (126) \times \frac{2^5}{x^5} \times 1 \times a^4 = 126 \times \frac{32}{x^5} a^4 \end{aligned}$$

$$T_5 = 4032 \frac{a^4}{x^5}$$

**(iv)  $y^3$  in the expansion of  $(x - \sqrt{y})^{11}$** 

$$a = x, x = (-\sqrt{y}), n = 11$$

We know that general term formula

$$\begin{aligned} T_{r+1} &= \binom{n}{r} a^{n-r} x^r \\ T_{r+1} &= \binom{11}{r} (x)^{11-r} (-\sqrt{y})^r \\ &= \binom{11}{r} x^{11-r} (-1)^r y^{r/2} \quad \dots\dots\dots (1) \end{aligned}$$

We have to find term involving  $y^3$ , so comparing the powers of  $y$  we get

$$\frac{r}{2} = 3 \Rightarrow r = 6 \text{ Put in (1)}$$

$$T_{6+1} = \binom{11}{6} x^{11-6} (-1)^6 y^{6/2}$$

$$T_7 = 462 x^5 \times 1 \times y^3$$

$$T_7 = 462 x^5 y^3$$



**Q.7 Find the coefficient of**

(i)  $x^5$  in the expansion of  $\left(x^2 - \frac{3}{2x}\right)^{10}$

(ii)  $x^n$  in the expansion of  $\left(x^2 - \frac{1}{x}\right)^{2n}$

**Solution:**

(i)  $x^5$  in the expansion of  $\left(x^2 - \frac{3}{2x}\right)^{10}$

**(Lahore Board 2003-04)**

We know that general term formula

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$n = 10, a = x^2, x = \left(-\frac{3}{2x}\right)$$

$$\begin{aligned}
 T_{r+1} &= \binom{10}{r} (x^2)^{10-r} \left(-\frac{3}{2x}\right)^r \\
 &= \binom{10}{r} (x)^{20-2r} \left(-\frac{3}{2}\right)^r \frac{1}{x^r} \\
 &= \binom{10}{r} (x)^{20-2r-r} \left(-\frac{3}{2}\right)^r \\
 &= \binom{10}{r} (x)^{20-3r} \left(-\frac{3}{2}\right)^r \dots\dots\dots (1)
 \end{aligned}$$

we have to find the coefficient of  $x^5$ , so comparing the powers of  $x$ , we get

$$20 - 3r = 5$$

$$15 = 3r \Rightarrow \boxed{r = 5}$$

Put in (1)

$$T_{5+1} = \binom{10}{5} x^{20-15} \left(-\frac{3}{2}\right)^5$$

$$T_6 = 252 \times x^5 \times \frac{-243}{32} = \frac{-15309}{8} x^5$$

$$\text{Coefficient of } x^5 \text{ is } \frac{-15309}{8}$$



(ii)  $x^n$  in the expansion of  $\left(x^2 - \frac{1}{x}\right)^{2n}$

We know that general term formula is

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$\begin{aligned} T_{r+1} &= \binom{2n}{r} (x^2)^{2n-r} \left(-\frac{1}{x}\right)^r \\ &= \binom{2n}{r} (x)^{4n-2r} \frac{(-1)^r}{x^r} \\ &= \binom{2n}{r} x^{4n-3r} (-1)^r \dots\dots\dots (1) \end{aligned}$$

we have to find the coefficient of  $x^n$ , so comparing the powers of  $x$ , we get

$$4n - 3r = n$$

$$4n - n = 3r$$

$$3n = 3r$$

$$\boxed{n = r}$$

Put in (1)

$$\begin{aligned} T_{n+1} &= \binom{2n}{n} x^{4n-3n} (-1)^n = \frac{(2n)!}{n! (2n-n)!} x^n (-1)^n \\ &= \frac{(2n)!}{n! n!} x^n (-1)^n \end{aligned}$$

$$\text{Coefficient of } x^n \text{ is } \frac{(-1)^n (2n)!}{(n!)^2}$$

**Q.8 Find 6<sup>th</sup> term in the expansion of  $\left(x^2 - \frac{3}{2x}\right)^{10}$**

**Solution:**

$$a = x^2, \quad x = \frac{-3}{2x}, \quad n = 10, \quad r = 5$$

We know by general term formula

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$T_{5+1} = \binom{10}{5} (x^2)^{10-5} \left(\frac{-3}{2x}\right)^5$$

$$T_6 = 252 \times x^{10} \times \frac{-243}{32 x^5}$$

$$T_6 = \frac{-15309}{8} x^5$$



**Q.9 Find the term independent of  $x$  in the following expansions.**

(i)  $\left(x - \frac{2}{x}\right)^{10}$  (Gujranwala Board 2003, Lahore Board 2008)

(ii)  $\left(\sqrt{x} + \frac{1}{2x^2}\right)^{10}$

(iii)  $(1 + x^2)^3 \left(1 + \frac{1}{x^2}\right)^4$

**Solution:**

(i)  $\left(x - \frac{2}{x}\right)^{10}$

$a = x, \quad x = \frac{-2}{x} \quad n = 10, \quad r = ?$

We know that general term formula is

$$\begin{aligned} T_{r+1} &= \binom{n}{r} a^{n-r} x^r \\ &= \binom{10}{r} x^{10-r} \left(\frac{-2}{x}\right)^r \\ &= \binom{10}{r} x^{10-r} \frac{(-2)^r}{x^r} \\ &= \binom{10}{r} x^{10-r-r} (-2)^r \\ &= \binom{10}{r} x^{10-2r} (-2)^r \dots\dots\dots (1) \end{aligned}$$

We have to find the term independent of  $x$  i.e.,  $x^0$  so comparing the powers of  $x$ , we have

$$10 - 2r = 0$$

$$10 = 2r \Rightarrow \boxed{r = 5} \quad \text{Put in (1)}$$

$$T_{5+1} = \binom{10}{5} x^{10-2(5)} (-2)^5$$

$$T_6 = 252 \times x^{10-10} \times (-32) = -8064 x^0 = -8064$$

Equating Index of  $x$  to 0 to get expression independent of  $x$



$$(ii) \quad \left( \sqrt{x} + \frac{1}{2x^2} \right)^{10}$$

$$a = \sqrt{x}, \quad x = \frac{1}{2x^2}, \quad n = 10, \quad r = ?$$

We know that general term formula

$$\begin{aligned} T_{r+1} &= \binom{n}{r} a^{n-r} x^r \\ &= \binom{10}{r} (\sqrt{x})^{10-r} \left( \frac{1}{2x^2} \right)^r \\ &= \binom{10}{r} x^{\frac{10-r}{2}} \frac{1}{2^r x^{2r}} \\ &= \binom{10}{r} x^{\frac{10-r}{2} - 2r} \left( \frac{1}{2^r} \right) \quad \dots\dots\dots (1) \end{aligned}$$

We have to find independent of  $x$  i.e.,  $x^0$  so comparing the powers of ' $x$ ', we get

$$\frac{10-r}{2} - 2r = 0$$

$$10 - r - 4r = 0$$

$$10 - 5r = 0$$

$$10 = 5r$$

$$2 = r \quad \text{Put in (1)}$$

$$\begin{aligned} T_{2+1} &= \binom{10}{2} x^{\frac{10-2}{2} - 2(2)} \left( \frac{1}{2^2} \right) \\ &= 45 x^{4-4} \left( \frac{1}{4} \right) = \frac{45}{4} x^0 = \frac{45}{4} \end{aligned}$$

$$(iii) \quad (1+x^2)^3 \left( 1 + \frac{1}{x^2} \right)^4$$

$$\begin{aligned} (1+x^2)^3 \left( 1 + \frac{1}{x^2} \right)^4 &= (1+x^2)^3 \frac{(1+x^2)^4}{x^8} \\ &= \frac{1}{x^8} (1+x^2)^7 \quad \dots\dots\dots (1) \end{aligned}$$



Now  $(1 + x^2)^7$ , we have  $a = 1$ ,  $x = x^2$ ,  $n = 7$ ,  $r = ?$

We know that general term formula is

$$\begin{aligned} T_{r+1} &= \binom{n}{r} a^{n-r} x^r = \binom{7}{r} (1)^{7-r} (x^2)^r \\ &= \binom{7}{r} x^{2r} \quad \text{equation (1) becomes} \\ &= \frac{1}{x^8} \binom{7}{r} x^{2r} \\ &= \left(\frac{7}{r}\right) x^{2r-8} \quad \dots\dots\dots (2) \end{aligned}$$

We have to find term independent of  $x$ .

i.e.,  $x^0$  so, comparing the powers of  $x$ .

$$2r - 8 = 0$$

$$2r = 8 \Rightarrow r = 4 \text{ put in (2).}$$

$$\begin{aligned} \binom{7}{4} x^{8-8} &= \frac{7!}{4! \times (7-4)!} x^0 \\ &= \frac{7!}{4! \times 3!} \\ &= \frac{7 \times 6 \times 5 \times 4!}{4! \times 3 \times 2 \times 1} = 35 \end{aligned}$$

### MIDDLE TERM

- (1) If  $n$  is even then  $\left(\frac{n}{2} + 1\right)^{\text{th}}$  term will be only one middle term.
- (2) If  $n$  is odd then  $\left(\frac{n+1}{2}\right)^{\text{th}}$  and  $\left(\frac{n+3}{2}\right)^{\text{th}}$  terms will be the two middle terms.

**Q.10 Determine the middle term in the following expansions.**

- (i)  $\left(\frac{1}{x} - \frac{x^2}{2}\right)^{12}$                       (ii)  $\left(\frac{3}{2}x - \frac{1}{3x}\right)^{11}$
- (iii)  $\left(2x - \frac{1}{2x}\right)^{2m+1}$

**Solution:**

(i)  $\left(\frac{1}{x} - \frac{x^2}{2}\right)^{12}$

Since  $n = 12$  is even so  $\left(\frac{n}{2} + 1\right)^{\text{th}}$  term i.e.,



$$\left(\frac{12}{2} + 1\right)^{\text{th}} \text{ term} = 7^{\text{th}} \text{ term is the middle term}$$

Thus  $r = 6$ . Also  $a = \frac{1}{x}$ ,  $x = \left(\frac{-x^2}{2}\right)$ ,  $n = 12$

We know that the general formula is

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$\begin{aligned} T_{6+1} &= \binom{12}{6} \left(\frac{1}{x}\right)^{12-6} \left(\frac{-x^2}{2}\right)^6 \\ &= 924 \frac{1}{x^6} \frac{x^{12}}{64} \end{aligned}$$

$$T_7 = \frac{231}{16} x^6$$

(ii)  $\left(\frac{3}{2}x - \frac{1}{3x}\right)^{11}$

Since  $n = 11$  is odd so  $\left(\frac{11+1}{2}\right)^{\text{th}}$  term and  $\left(\frac{11+3}{2}\right)^{\text{th}}$  term i.e.,  $6^{\text{th}}$  &  $7^{\text{th}}$  terms will be the two middle terms.

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$T_{5+1} = \binom{11}{5} \left(\frac{3}{2}x\right)^{11-5} \left(-\frac{1}{3x}\right)^5$$

$$\begin{aligned} T_6 &= 462 \times \left(\frac{3}{2}x\right)^6 \frac{(-1)^5}{(3x)^5} \\ &= 462 \times \frac{(3x)^{6-5}}{64} \times -1 \\ &= \frac{-462 \times 3x}{64} = \frac{-693x}{32} \end{aligned}$$

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$\begin{aligned} T_{6+1} &= \binom{11}{6} \left(\frac{3}{2}x\right)^{11-6} \left(-\frac{1}{3x}\right)^6 \\ &= 462 \times \frac{(3x)^5}{(2)^5} \times \frac{1}{(3x)^6} \end{aligned}$$



$$= 462 \times \frac{1}{32} \times \frac{1}{(3x)^{6-5}}$$

$$T_6 = \frac{462}{32 \times 3x} = \frac{77}{16x}$$

Hence two middle terms are  $-\frac{693x}{32}$  and  $\frac{77}{16x}$

(iii)  $\left(2x - \frac{1}{2x}\right)^{2m+1}$

As  $2m+1$  is odd, so there are two middle terms i.e.,  $\left(\frac{2m+1+1}{2}\right)$  and  $\left(\frac{2m+1+3}{2}\right)$  are two middle terms.

$(m+1)^{\text{th}}$  and  $(m+2)^{\text{th}}$  terms

For  $(m+1)^{\text{th}}$  term

$$r = m, \quad n = 2m+1, \quad a = 2x, \quad x = \left(-\frac{1}{2x}\right)$$

$$\begin{aligned} T_{m+1} &= \binom{2m+1}{m} (2x)^{2m+1-m} \left(-\frac{1}{2x}\right)^m \\ &= \frac{(2m+1)!}{m! [2m+1-m]!} (2x)^{m+1-m} (-1)^m \\ &= \frac{(2m+1)!}{m! (m+1)!} 2x (-1)^m \end{aligned}$$

$$T_{m+1} = 2(-1)^m \frac{(2m+1)!}{m! (m+1)!} x$$

For  $(m+2)^{\text{th}}$  term

$$r = m+2-1 = m+1$$

$$n = 2m+1, \quad a = 2x, \quad x = \left(-\frac{1}{2x}\right)$$

$$\begin{aligned} T_{r+1} &= \binom{n}{r} a^{n-r} x^r \\ &= \binom{2m+1}{m+1} (2x)^{2m+1-m-1} \left(-\frac{1}{2x}\right)^{m+1} \\ &= \frac{(2m+1)!}{(m+1)! [(2m+1-m-1)]!} (2x)^m \frac{(-1)^{m+1}}{(2x)^{m+1}} = \frac{(2m+1)!}{(m+1)! (m)!} \frac{(-1)^{m+1}}{(2x)^{m+1-m}} \\ T_{m+2} &= \frac{(2m+1)! (-1)^{m+1}}{m! (m+1)! 2x} \end{aligned}$$

$T_{m+1}$  and  $T_{m+2}$  are two middle terms.



**Q.11 Find  $(2n + 1)^{\text{th}}$  term from the end in the expansion of  $\left(x - \frac{1}{2x}\right)^{3n}$**

**Solution:**

To find  $(2n + 1)^{\text{th}}$  terms, we have  $r = 2n$

And for the term from the end, we have

$$a = -\frac{1}{2x} \text{ and } x = x$$

By general term formula

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$\begin{aligned} T_{2n+1} &= \binom{3n}{2n} \left(-\frac{1}{2x}\right)^{3n-2n} (x)^{2n} \\ &= \frac{3n!}{2n! (3n-2n)!} \left(\frac{-1}{2x}\right)^n x^{2n} \\ &= \frac{3n!}{(2n)! n!} \frac{(-1)^n}{2^n x^n} x^{2n} \\ &= \frac{3n! (-1)^n}{2n! n! 2^n} x^{2n-n} \\ &= \frac{(3n)! (-1)^n x^n}{2n! n! 2n} \end{aligned}$$

**Q.12 Show that middle term of  $(1+x)^{2n}$  is  $\frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{n!} 2^n \cdot x^n$**

**Solution:**

As  $2n$  is even so  $\left(\frac{2n}{2} + 1\right)^{\text{th}}$  term is the middle term i.e.,  $(n+1)^{\text{th}}$  term  $r = n$

General term formula is

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$\begin{aligned} T_{n+1} &= \binom{2n}{n} (1)^{2n-n} x^n \\ &= \frac{(2n)!}{n! [2n-n]!} (1)^n x^n = \frac{(2n)!}{n! n!} x^n \\ &= \frac{(2n)(2n-1)(2n-2)(2n-3)(2n-4) \dots 5 \times 4 \times 3 \times 2 \times 1}{n! n!} x^n \end{aligned}$$



$$\begin{aligned}
 &= \frac{[(2n)(2n-2)(2n-4) \dots 4 \times 2][(2n-1)(2n-3)(2n-5) \dots 5 \times 3 \times 1] x^n}{n! n!} \\
 &= \frac{[2^n (n)(n-1)(n-2) \dots (n-2) \dots \times 2 \times 1][(2n-1)(2n-3) \dots 5 \times 3 \times 1] x^n}{n! n!} \\
 &= \frac{2^n n! [(2n-1)(2n-3) \dots 5 \times 3 \times 1] x^n}{n! n!} \\
 T_{n+1} &= \frac{2^n [1 \times 3 \times 5 \times \dots (2n-3)(2n-1)] x^n}{n!}
 \end{aligned}$$

**Q.13 Show that:**  $\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots + \binom{n}{n-1} = 2^{n-1}$

**Solution:**

We know that

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n \quad (1)$$

Put  $x = 1$  in equation (1)

$$\begin{aligned}
 (1+1)^n &= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \binom{n}{4} + \dots + \binom{n}{n-1} + \binom{n}{n} \\
 2^n &= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \binom{n}{4} + \dots + \binom{n}{n-1} + \binom{n}{n} \quad (2)
 \end{aligned}$$

Next put  $x = -1$  in equation (1)

$$(1-1)^n = \binom{n}{0} - \binom{n}{1}(-1) + \binom{n}{2}(-1)^2 + \binom{n}{3}(-1)^3 + \dots + \binom{n}{n-1}(-1)^{n-1} + \binom{n}{n}(-1)^n$$

if  $n$  is even then

$$\begin{aligned}
 0 &= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots + (-1)^{n-1} \binom{n}{n-1} + (-1)^n \binom{n}{n} \\
 \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots + \binom{n}{n} &= \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots + \binom{n}{n-1} \quad (3)
 \end{aligned}$$

We can write (2) as.

$$2^n = \left\{ \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots + \binom{n}{n} \right\} + \left\{ \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots + \binom{n}{n-1} \right\} \quad (4)$$

Using (3) in (4)

$$\begin{aligned}
 2^n &= \left\{ \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots + \binom{n}{n-1} \right\} + \left\{ \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots + \binom{n}{n-1} \right\} \\
 2^n &= 2 \left\{ \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots + \binom{n}{n-1} \right\} \\
 \frac{2^n}{2} &= \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots + \binom{n}{n-1} \\
 \Rightarrow \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots + \binom{n}{n-1} &= 2^{n-1}
 \end{aligned}$$



**Q.14 Show that**  $\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \frac{1}{4}\binom{n}{3} + \dots + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1}-1}{n+1}$

**Solution:**

$$\begin{aligned} \text{L. H. S.} &= \binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \frac{1}{4}\binom{n}{3} + \dots + \frac{1}{n+1}\binom{n}{n} \\ &= \frac{n!}{0!(n-0)!} + \frac{1}{2} \frac{n!}{1!(n-1)!} + \frac{1}{3} \frac{n!}{2!(n-2)!} + \frac{1}{4} \frac{n!}{3!(n-3)!} + \dots + \frac{1}{n+1} \frac{n!}{n!(n-n)!} \end{aligned}$$

Taking common  $n!$

$$= n! \left[ \frac{1}{n!} + \frac{1}{2!(n-1)!} + \frac{1}{3!(n-2)!} + \frac{1}{4!(n-3)!} + \dots + \frac{1}{(n+1)n!} \right]$$

Now multiplying and dividing by  $n+1$

$$\begin{aligned} &= \frac{(n+1)n!}{(n+1)} \left[ \frac{1}{n!} + \frac{1}{2!(n-1)!} + \frac{1}{3!(n-2)!} + \frac{1}{4!(n-3)!} + \dots + \frac{1}{(n+1)n!} \right] \\ &= \frac{(n+1)!}{n+1} \left[ \frac{1}{n!} + \frac{1}{2!(n-1)!} + \frac{1}{3!(n-2)!} + \frac{1}{4!(n-3)!} + \dots + \frac{1}{(n+1)n!} \right] \\ &= \frac{1}{n+1} \left[ \frac{(n+1)!}{n!} + \frac{(n+1)!}{2!(n-1)!} + \frac{(n+1)!}{3!(n-2)!} + \frac{(n+1)!}{4!(n-3)!} + \dots + \frac{(n+1)!}{(n+1)n!} \right] \\ &= \frac{1}{n+1} \left[ \binom{n+1}{1} + \binom{n+1}{2} + \binom{n+1}{3} + \binom{n+1}{4} + \dots + \binom{n+1}{n+1} \right] \end{aligned}$$

Adding and subtracting  $\binom{n+1}{0}$

$$= \frac{1}{n+1} \left[ \binom{n+1}{0} + \binom{n+1}{1} + \binom{n+1}{2} + \binom{n+1}{3} + \dots + \binom{n+1}{n+1} - \binom{n+1}{0} \right] \quad (1)$$

We know that

$$\binom{n+1}{0} + \binom{n+1}{1} + \binom{n+1}{2} + \binom{n+1}{3} + \dots + \binom{n+1}{n+1} = 2^{n+1}$$

and  $\binom{n+1}{0} = 1$

So (1) becomes

$$\frac{1}{n+1} [2^{n+1} - 1] = \frac{2^{n+1}-1}{n+1} = \text{R.H.S.}$$

Hence proved.



## EXERCISE 8.3

### BINOMIAL SERIES

(Lahore Board 2009, 11)

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

Here index  $n$  is negative integer or a fraction.

**Q.1** Expand the following upto 4 terms, taking value of  $x$  such that the expansion in each case is valid.

- |                                       |                                        |
|---------------------------------------|----------------------------------------|
| (i) $(1-x)^{1/2}$                     | (ii) $(1+2x)^{-1}$                     |
| (iii) $(1+x)^{-1/3}$                  | (iv) $(4-3x)^{1/2}$                    |
| (v) $(8-2x)^{-1}$ (Lahore Board 2008) | (vi) $(2-3x)^{-2}$ (Lahore Board 2010) |
| (vii) $\frac{(1-x)^{-1}}{(1+x)^2}$    | (viii) $\frac{\sqrt{1+2x}}{1-x}$       |
| (ix) $\frac{(4+2x)^{1/2}}{(2-x)}$     | (x) $(1+x-2x^2)^{1/2}$                 |
| (xi) $(1-2x+3x^2)^{-1/3}$             |                                        |

**Solution:**

(i)  $(1-x)^{1/2}$

By binomial series

$$\begin{aligned}
 &= \left( 1 + \frac{1}{2}(-x) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-x)^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(-x)^3 + \dots \right) \\
 &= 1 - \frac{1}{2}x + \frac{1}{2}\left(-\frac{1}{2}\right) \times \frac{1}{2}x^2 + \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) \times \frac{1}{6}(-x^3) + \dots \\
 &= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 + \dots
 \end{aligned}$$

Valid if  $|x| < 1$

(ii)  $(1+2x)^{-1}$

$$\begin{aligned}
 &1 + (-1)(2x) + \frac{(-1)(-1-1)}{2!}(2x)^2 + \frac{(-1)(-1-1)(-1-2)}{3!}(2x)^3 + \dots \\
 &= 1 - 2x + 4x^2 - 8x^3 + \dots
 \end{aligned}$$

Valid if  $|2x| < 1$

$$2|x| < 1$$

$$\Rightarrow |x| < \frac{1}{2}$$



(iii)  $(1+x)^{-1/3}$

$$1 + \left(-\frac{1}{3}\right)x + \frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)}{2!}x^2 + \frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)\left(-\frac{1}{3}-2\right)}{3!}x^3 + \dots$$

$$= 1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{14}{81}x^3 + \dots$$

Valid if  $|x| < 1$

(iv)  $(4-3x)^{1/2}$

$$(4)^{1/2} \left(1 - \frac{3x}{4}\right)^{1/2}$$

$$= 2 \left[ 1 + \frac{1}{2} \left(-\frac{3x}{4}\right) + \frac{\frac{1}{2} \left(\frac{1}{2}-1\right)}{2!} \left(-\frac{3x}{4}\right)^2 + \frac{\frac{1}{2} \left(\frac{1}{2}-1\right) \left(\frac{1}{2}-2\right)}{3!} \left(-\frac{3x}{4}\right)^3 + \dots \right]$$

$$= 2 \left[ 1 - \frac{3x}{8} + \frac{\frac{1}{2} \left(-\frac{1}{2}\right)}{2} \times \frac{9x^2}{16} + \frac{\frac{1}{2} \left(-\frac{1}{2}\right) \left(-\frac{3}{2}\right)}{6} \left(-\frac{27x^3}{64}\right) + \dots \right]$$

$$= 2 - \frac{3x}{4} - \frac{9}{64}x^2 - \frac{27}{512}x^3 + \dots$$

Expansion is valid if

$$\left| \frac{3}{4}x \right| < 1$$

$$\Rightarrow \frac{3}{4}|x| < 1$$

$$\Rightarrow |x| < \frac{4}{3}$$

(v)  $(8-2x)^{-1}$

(Lahore Board 2008)

$$= 8^{-1} \left(1 - \frac{2x}{8}\right)^{-1}$$

$$= \frac{1}{8} \left[ 1 - \frac{x}{4} \right]^{-1}$$

$$= \frac{1}{8} \left[ 1 + \frac{1}{4}x + \frac{-1 \times -2}{2 \times 1} \frac{1}{16}x^2 + \frac{(-1) \times (-2) \times (-3)}{3 \times 2 \times 1} \times \frac{-1}{64}x^3 + \dots \right]$$

$$= \frac{1}{8} \left[ 1 + \frac{1}{4}x + \frac{1}{16}x^2 + \frac{1}{64}x^3 + \dots \right]$$

$$= \frac{1}{8} + \frac{1}{32}x + \frac{1}{128}x^2 + \frac{1}{512}x^3 + \dots$$



The expansion valid only if

$$\left| \frac{x}{4} \right| < 1$$

$$\Rightarrow \frac{1}{4}|x| < 1$$

$$\Rightarrow |x| < 4$$

(vi)  $(2 - 3x)^{-2}$

(Lahore Board 2010)

$$2^{-2} \left( 1 - \frac{3x}{2} \right)^{-2}$$

$$= \frac{1}{4} \left[ 1 + (-2) \left( \frac{-3}{2} x \right) + \frac{(-2)(-2-1)}{2!} \left( \frac{-3x}{2} \right)^2 + \frac{(-2)(-2-1)(-2-2)}{3!} \left( \frac{-3}{2} x \right)^3 + \dots \right]$$

$$= \frac{1}{4} \left[ 1 + 3x + \frac{-2x-3}{2} \times \frac{9x^2}{4} + \frac{(-2)(-3)(-4)}{6} \times \frac{-27x^3}{8} + \dots \right]$$

$$= \frac{1}{4} \left[ 1 + 3x + \frac{27x^2}{4} + \frac{27x^3}{2} + \dots \right]$$

$$= \frac{1}{4} + \frac{3}{4}x + \frac{27x^2}{16} + \frac{27x^3}{8} + \dots$$

The above expansion is valid only if

$$\left| \frac{3x}{2} \right| < 1$$

$$\Rightarrow \frac{3}{2}|x| < 1$$

$$\Rightarrow |x| < \frac{2}{3}$$

(vii)  $\frac{(1-x)^{-1}}{(1+x)^2}$

$$= (1-x)^{-1} (1+x)^{-2}$$

$$= \left[ 1 + x + \frac{(-1)(-1-1)}{2!} (-x)^2 + \frac{(-1)(-1-1)(-1-2)}{3!} (-x)^3 + \dots \right]$$

$$\left[ 1 - 2x + \frac{(-2)(-2-1)}{2!} x^2 + \frac{(-2)(-2-1)(-2-2)}{3!} (+x)^3 + \dots \right]$$

$$= \left[ 1 + x + \frac{(-1)(-2)}{2} (x)^2 + \frac{(-1)(-2)(-3)}{6} (-x^3) + \dots \right]$$

$$\left[ 1 - 2x + \frac{(-2)(-3)}{2} x^2 + \frac{(-2)(-3)(-4)}{6} x^3 + \dots \right]$$



$$\begin{aligned}
&= [1 + x + x^2 + x^3 + \dots] [1 - 2x + 3x^2 - 4x^3 + \dots] \\
&= 1 - 2x + 3x^2 - 4x^3 + x - 2x^2 + 3x^3 + x^2 - 2x^3 + x^3 + \dots \\
&= 1 - x + 2x^2 - 2x^3 + \dots
\end{aligned}$$

The above expansion are valid if

$$|x| < 1$$

(viii)  $\frac{\sqrt{1+2x}}{1-x}$

$$\begin{aligned}
&(1+2x)^{1/2} (1-x)^{-1} \\
&= (1-x)^{-1} (1+2x)^{1/2} \\
&= \left[ 1 + x + \frac{(-1)(-1-1)}{2!} (-x)^2 + \frac{(-1)(-1-1)(-1-2)}{3!} (-x)^3 + \dots \right] \\
&\quad \left[ 1 + \frac{1}{2} 2x + \frac{\frac{1}{2} \left( \frac{1}{2} - 1 \right)}{2!} (2x)^2 + \frac{\frac{1}{2} \left( \frac{1}{2} - 1 \right) \left( \frac{1}{2} - 2 \right)}{3!} (2x)^3 + \dots \right] \\
&= \left[ 1 + x + \frac{-1 \times -2}{2} x^2 + \frac{-1 \times -2 \times -3}{6} (-x^3) + \dots \right] \\
&\quad \left[ 1 + x + \frac{\frac{1}{2} \left( -\frac{1}{2} \right)}{2} 4x^2 + \frac{\frac{1}{2} \left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right)}{6} 8x^3 + \dots \right] \\
&= [1 + x + x^2 + x^3 + \dots] \left[ 1 + x - \frac{1}{2} x^2 + \frac{1}{2} x^3 + \dots \right] \\
&= 1 + x - \frac{1}{2} x^2 + \frac{1}{2} x^3 + x + x^2 - \frac{1}{2} x^3 + x^2 + x^3 + x^3 + \dots \\
&= 1 + 2x + \frac{3}{2} x^2 + 2x^3 + \dots
\end{aligned}$$

The above expansion valid if

$$|x| < \frac{1}{2} \quad \text{and} \quad |x| < 1$$

(ix)  $\frac{(4+2x)^{1/2}}{(2-x)}$

$$\begin{aligned}
&(4+2x)^{1/2} (2-x)^{-1} \\
&= 4^{1/2} \left( 1 + \frac{2}{4} x \right)^{1/2} 2^{-1} \left( 1 - \frac{x}{2} \right)^{-1}
\end{aligned}$$



$$\begin{aligned}
&= 2 \left( 1 + \frac{1}{2} \frac{x}{2} + \frac{1}{2} \frac{\frac{1}{2} \left( \frac{1}{2} - 1 \right)}{2!} \frac{x^2}{4} + \frac{\frac{1}{2} \left( \frac{1}{2} - 1 \right) \left( \frac{1}{2} - 2 \right)}{3!} \left( \frac{x}{2} \right)^3 + \dots \right) \\
&\quad \frac{1}{2} \left[ 1 + \frac{x}{2} + \frac{(-1)(-1-1)}{2!} \left( -\frac{x}{2} \right)^2 + \frac{(-1)(-1-1)(-1-2)}{3!} \left( -\frac{x}{2} \right)^3 + \dots \right] \\
&= \left[ 1 + \frac{1}{4}x - \frac{1}{32}x^2 + \frac{1}{128}x^3 + \dots \right] \left[ 1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \dots \right] \\
&= 1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \frac{1}{4}x + \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{1}{32}x^2 - \frac{1}{64}x^3 + \frac{1}{128}x^3 \\
&= 1 + \frac{3}{4}x + \frac{11}{32}x^2 + \frac{23}{128}x^3 + \dots
\end{aligned}$$

The expansion of  $\left(1 + \frac{x}{2}\right)^{1/2}$  and  $\left(1 - \frac{x}{2}\right)^{-1}$  are valid if

$$\left| \frac{x}{2} \right| < 1$$

$$\Rightarrow |x| < 2$$

(x)  $(1 + x - 2x^2)^{1/2}$

$$\begin{aligned}
&= 1 + \frac{1}{2}(x - 2x^2) + \frac{\frac{1}{2} \left( \frac{1}{2} - 1 \right)}{2!} (x - 2x^2)^2 + \frac{\frac{1}{2} \left( \frac{1}{2} - 1 \right) \left( \frac{1}{2} - 2 \right)}{3!} (x - 2x^2)^3 + \dots \\
&= 1 + \frac{1}{2}(x - 2x^2) + \frac{\frac{1}{2} \times -\frac{1}{2}}{2} (x^2 + 4x^4 - 4x^3) + \frac{\frac{1}{2} \left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right)}{6} (x^3 - 8x^6 - 6x^4 + 12x^5) + \dots \\
&= 1 + \frac{1}{2}x - x^2 - \frac{1}{8}x^2 - \frac{1}{2}x^4 + \frac{1}{2}x^3 + \frac{1}{16}x^3 - \frac{3}{8}x^4 + \frac{3}{4}x^5 - \frac{1}{2}x^6 + \dots \\
&= 1 + \frac{1}{2}x - \frac{9}{8}x^2 + \frac{9}{16}x^3 + \dots
\end{aligned}$$

The above expansion is valid only if  $|x - 2x^2| < 1$  that is either

$$\begin{aligned}
&x - 2x^2 < 1 \quad \text{or} \quad -(x - 2x^2) < 1 \\
\Rightarrow -2x^2 + x - 1 < 0 \quad \dots\dots\dots (1) \quad &2x^2 - x - 1 < 0 \quad \dots\dots\dots (2) \\
2x^2 - 2x + x - 1 < 0 \\
(x - 1)(2x + 1) < 0 \\
-\frac{1}{2} < x < 1
\end{aligned}$$



$$\begin{aligned}
 \text{(xi)} \quad & (1 - 2x + 3x^2)^{-1/3} \\
 & [1 + (3x^2 - 2x)]^{-1/3} \\
 & = \left[ 1 + \left(\frac{-1}{3}\right)(3x^2 - 2x) + \frac{\left(-\frac{1}{3}\right)\left(\frac{-1}{3} - 1\right)}{2!} (3x^2 - 2x)^2 + \frac{\frac{-1}{3}\left(\frac{-1}{3} - 1\right)\left(\frac{-1}{3} - 2\right)}{3!} (3x^2 - 2x)^3 + \dots \right] \\
 & = 1 - \frac{1}{3}(3x^2 - 2x) - \frac{1}{3} \times \frac{-4}{3} \times \frac{1}{2} (9x^4 + 4x^2 - 12x^3) + \frac{-1}{3} \times \frac{-4}{3} \times \frac{-7}{2} \times \frac{1}{6} \\
 & \quad (27x^6 - 8x^3 - 54x^5 + 36x^4) + \dots \\
 & = 1 - x^2 + \frac{2}{3}x + \frac{2}{9}(9x^4 + 4x^2 - 12x^3) - \frac{7}{27}(27x^6 - 8x^3 - 54x^5 + 36x^4) + \dots \\
 & = 1 - x^2 + \frac{2}{3}x + 2x^4 + \frac{8}{9}x^2 - \frac{24}{9}x^3 - 7x^6 + \frac{56}{27}x^3 + 14x^5 + \dots \\
 & = 1 + \frac{2}{3}x - \frac{1}{9}x^2 - \frac{16}{27}x^3 + \dots
 \end{aligned}$$

The above expansion is valid only if

$$\begin{aligned}
 |3x^2 - 2x| &< 1 \\
 3x^2 - 2x &< 1 \quad - (3x^2 - 2x) < 1 \\
 3x^2 - 2x - 1 &< 1 \\
 3x^2 - 3x + x - 1 &< 1 \\
 (3x + 1)(x - 1) &< 1 \\
 \frac{-1}{3} &< x < 1
 \end{aligned}$$

**Q.2 Using Binomial theorem find the value of the following to three places of decimals.**

**Solution:**

$$\begin{aligned}
 \text{(i)} \quad & \sqrt{99} \\
 & = (99)^{1/2} = (100 - 1)^{1/2} = (100)^{1/2} \left(1 - \frac{1}{100}\right)^{1/2} \\
 & = 10 \left[ 1 + \left(\frac{1}{2}\right)\left(\frac{-1}{100}\right) + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!} \left(\frac{-1}{100}\right)^2 + \dots \right] \\
 & = 10 \left[ 1 - \frac{1}{2 \times 100} + \frac{1}{2} \times \frac{-1}{2} \times \frac{1}{2} \times \frac{1}{100 \times 100} + \dots \right]
 \end{aligned}$$



$$= 10 - \frac{1}{20} - \frac{1}{8000} + \dots$$

$$= 10 - 0.05 - 0.000125 + \dots = 9.950$$

(ii)  $(0.98)^{1/2}$

$$= (1 - .02)^{1/2}$$

$$= 1 + \frac{1}{2}(-.02) + \frac{1}{2}\left(\frac{1}{2} - 1\right)(-.02)^2 + \dots$$

$$= 1 - .01 + \frac{1}{2}\left(-\frac{1}{2}\right)(.0004) + \dots$$

$$= 1 - .01 - .00005 + \dots = .990$$

(iii)  $(1.03)^{1/3}$

$$= (1 + .03)^{1/3}$$

$$= \left( 1 + \frac{1}{3}(.03) + \frac{\frac{1}{3}\left(\frac{1}{3} - 1\right)}{2!}(.03)^2 + \dots \right)$$

$$= 1 + .01 + \frac{1}{3} \times \frac{-2}{3} \times \frac{1}{2} \times .0009 + \dots$$

$$= 1 + .01 - .0001 + \dots = 1.010$$

(iv)  $\sqrt[3]{65}$

$$= (65)^{1/3} = (64 + 1)^{1/3}$$

$$= (64)^{1/3} \left( 1 + \frac{1}{64} \right)^{1/3}$$

$$= 4 \left[ 1 + \frac{1}{3}\left(\frac{1}{64}\right) + \frac{\frac{1}{3}\left(\frac{1}{3} - 1\right)}{2!}\left(\frac{1}{64}\right)^2 + \dots \right]$$

$$= 4 \left[ 1 + \frac{1}{192} + \frac{\frac{1}{3}\left(\frac{-2}{3}\right)}{2} \times \frac{1}{4096} + \dots \right]$$

$$= 4 + 0.021 - 0.0001 + \dots = 4.021$$

(v)  $\sqrt[4]{17}$

$$= (17)^{1/4} = (16 + 1)^{1/4} = 16^{1/4} \left( 1 + \frac{1}{16} \right)^{1/4}$$



$$\begin{aligned}
 &= 2^{4 \times (1/4)} \left[ 1 + \frac{1}{4} \left( \frac{1}{16} \right) + \dots \right] \\
 &= 2 + \frac{1}{2 \times 16} + \dots = 2 + \frac{1}{32} + \dots \\
 &= 2 + 0.031 + \dots = 2.031
 \end{aligned}$$

(vi)  $\sqrt[5]{31}$

$$\begin{aligned}
 &= (31)^{1/5} = (32 - 1)^{1/5} \\
 &= 32^{1/5} \left( 1 - \frac{1}{32} \right)^{1/5} \\
 &= 2^{5 \times (1/5)} \left[ 1 + \left( \frac{1}{5} \right) \left( \frac{-1}{32} \right) + \dots \right] \\
 &= 2 - \frac{1}{5 \times 16} + \dots = 2 - 0.013 = 1.987
 \end{aligned}$$

(vii)  $\frac{1}{\sqrt[3]{998}}$

$$\begin{aligned}
 &= \left( \frac{1}{(998)^{1/3}} \right) = (998)^{-1/3} \\
 &= (1000 - 2)^{-1/3} = (1000)^{-1/3} \left[ 1 - \frac{2}{1000} \right]^{-1/3} \\
 &= 10^{3 \times (-1/3)} \left[ 1 - \frac{1}{500} \right]^{-1/3} \\
 &= 10^{-1} \left[ 1 + \left( \frac{1}{3} \right) \left( \frac{1}{500} \right) + \dots \right] \\
 &= \frac{1}{10} \left[ 1 + \frac{1}{1500} + \dots \right] = \frac{1}{10} + \frac{1}{15000} + \dots \\
 &= 0.1 + 0.000067 + \dots = 0.1000
 \end{aligned}$$

(viii)  $\frac{1}{\sqrt[5]{252}}$

$$\begin{aligned}
 &= \frac{1}{(252)^{1/5}} = (252)^{-1/5} = (243 + 9)^{-1/5}
 \end{aligned}$$



$$\begin{aligned}
 &= 243^{-1/5} \left[ 1 + \frac{9}{243} \right]^{-1/5} = 3^{5 \times -1/5} \left[ 1 + \left( \frac{-1}{5} \right) \left( \frac{9}{243} \right) + \dots \right] \\
 &= 3^{-1} \left[ 1 - \frac{1}{5} \times \frac{1}{27} + \dots \right] = \frac{1}{3} \left[ 1 - \frac{1}{135} + \dots \right] \\
 &= \frac{1}{3} [1 - 0.007 + \dots] = \frac{1}{3} [0.993] = 0.331
 \end{aligned}$$

(ix)  $\frac{\sqrt{7}}{\sqrt{8}}$

$$\begin{aligned}
 &= \left( \frac{7}{8} \right)^{1/2} = \left( 1 - \frac{1}{8} \right)^{1/2} \\
 &= 1 + \left( \frac{1}{2} \right) \left( \frac{-1}{8} \right) + \dots \\
 &= 1 - \frac{1}{16} + \dots = 1 - 0.063 + \dots = 0.938
 \end{aligned}$$

(x)  $(.998)^{-1/3}$

$$\begin{aligned}
 &= (1 - 0.002)^{-1/3} \\
 &= 1 + \left( \frac{-1}{3} \right) (-0.002) + \frac{\left( \frac{-1}{3} \right) \left( \frac{-1}{3} - 1 \right)}{2 \times 1} (-0.002)^2 + \dots \\
 &= 1 + 0.001 + \frac{2}{9} (0) + \dots \\
 &= 1 + 0.001 + 0 + \dots = 1.001
 \end{aligned}$$

(xi)  $\frac{1}{\sqrt[6]{486}}$

$$\begin{aligned}
 &= \frac{1}{(486)^{1/6}} = (486)^{-1/6} = (729 - 243)^{-1/6} \\
 &= (729)^{-1/6} \times \left[ 1 - \frac{243}{729} \right]^{-1/6} = (3^6)^{-1/6} \left[ 1 - \frac{1}{3} \right]^{-1/6} \\
 &= \frac{1}{3} \left[ 1 - \frac{1}{3} \right]^{-1/6} \Rightarrow \frac{1}{3} \left[ 1 + \left( \frac{-1}{6} \right) \left( \frac{-1}{3} \right) + \frac{\left( \frac{-1}{6} \right) \left( \frac{-1}{6} - 1 \right)}{2!} \left( \frac{-1}{3} \right)^2 + \dots \right] \\
 &= \frac{1}{3} [1 + 0.0555 + 0.0108 + \dots] \\
 &= \frac{1}{3} [1.06895] = 0.356
 \end{aligned}$$



(xii)  $(1280)^{1/4}$

$$\begin{aligned}
 &= (1296 - 16)^{1/4} = 1296^{1/4} \left( 1 - \frac{16}{1296} \right)^{1/4} \\
 &= 6^{4 \times (1/4)} \left[ 1 + \left( \frac{1}{4} \right) \left( -\frac{16}{296} \right) + \dots \right] \\
 &= 6 \left[ 1 - \frac{1}{324} + \dots \right] = 6 [1 - 0.003 + \dots] = 6 [0.997] = 5.981
 \end{aligned}$$

**Q.3 Find the coefficient of  $x^n$  in the expansion**

(i)  $\frac{1+x^2}{(1+x)^2}$

(ii)  $\frac{(1+x)^2}{(1-x)^2}$

(iii)  $\frac{(1+x)^3}{(1-x)^2}$

(iv)  $\frac{(1+x)^2}{(1-x)^3}$

(v)  $(1-x+x^2-x^3+\dots)$  (Gujranwala Board 2005)

**Solution:**

(i)  $\frac{1+x^2}{(1+x)^2}$

$$\begin{aligned}
 &= (1+x^2)(1+x)^{-2} \\
 &= (1+x^2) \left[ 1 + (-2)(x) + \frac{(-2)(-2-1)x^2}{2!} + \frac{(-2)(-2-1)(-3-1)(x)^3}{3!} + \dots \right] \\
 &= (1+x^2) \left[ 1 + (-2)(x) + \frac{(-2)(-3)x^2}{2!} + \frac{(-2)(-3)(-4)}{3 \times 2 \times 1} x^3 + \dots \right] \\
 &= (1+x^2) [1 + (-1)2x + (-1)^2 3x^2 + (-1)^3 4x^3 + \dots \\
 &\quad + \dots + (-1)^{n-2} (n-1)^{n-2} x + (-1)^{n-1} n x^{n-1} + (-1)^n (n+1) x^n] \\
 &= (-1)^{n-2} (n-1) x^n + (-1)^n (n+1) x^n \\
 &= [(-1)^n (-1)^2 (n-1) + (-1)^n (n+1)] x^n = (-1)^n [n-1+n+1] x^n \\
 &= (-1)^n \cdot (2n) x^n
 \end{aligned}$$

Coefficient of  $x^n$  is,  $(-1)^n \times (2n)$

(ii)  $\frac{(1+x)^2}{(1-x)^2}$

$$= (1+x)^2 (1-x)^{-2}$$



$$\begin{aligned}
&= (1 + 2x + x^2) \left( 1 + 2x + \frac{(-2)(-2-1)(-x)^2}{2!} + \dots \right) \\
&= (1 + 2x + x^2) \left( 1 + 2x + \frac{(-2)(-3)}{2 \times 1} x^2 + \dots \right) \\
&= (1 + 2x + x^2) [1 + 2x + 3x^2 + \dots + (n-1)x^{n-2} + nx^{n-1} + (n+1)x^n]
\end{aligned}$$

Now multiplying the terms to get terms involving  $x^n$ .

$$\begin{aligned}
&= (n+1)x^n + 2nx^{n-1+1} + (n-1)x^{n-2+2} \\
&= (n+1)x^n + 2nx^n + (n-1)x^n \\
&= (n+1+2n+n-1)x^n \\
&= 4nx^n
\end{aligned}$$

Hence coefficient of  $x^n$  is  $4n$

(iii)  $\frac{(1+x)^3}{(1-x)^2}$

$$\begin{aligned}
&= (1+x)^3 (1-x)^{-2} \\
&= \left[ 1 + 3x + \frac{(3)(3-1)}{2!} (x)^2 + \frac{3(3-1)(3-2)}{3!} x^3 \right] \\
&\quad \left[ 1 + 2x + \frac{(-2)(-2-1)(-x)^2}{2!} + \frac{(-2)(-2-1)(-2-2)}{3!} (-x)^3 + \dots \right] \\
&= \left[ 1 + 3x + \frac{3 \times 2}{2 \times 1} x^2 + \frac{3 \times 2 \times 1}{3 \times 2 \times 1} x^3 \right] \\
&\quad \left[ 1 + 2x + \frac{(-2)(-3)}{2 \times 1} x^2 + \frac{(-2)(-3)(-4)}{3 \times 2 \times 1} (-x)^3 + \dots \right] \\
&= [1 + 3x + 3x^2 + x^3] \\
&\quad [1 + 2x + 3x^2 + 4x^3 + \dots + (n-2)x^{n-3} + (n-1)x^{n-2} + nx^{n-1} + (n+1)x^n] \\
&= (n+1)x^n + 3nx^{n-1+1} + 3(n-1)x^{n-2+2} + (n-2)x^{n-3+3} \\
&= (n+1)x^n + 3nx^n + 3(n-1)x^n + (n-2)x^n \\
&= (n+1+3n+3n-3+n-2)x^n \\
&= (8n-4) \cdot x^n = 4(2n-1)x^n
\end{aligned}$$

Hence coefficient of  $x^n$  is  $4(2n-1)$ .



$$\begin{aligned}
 \text{(iv)} \quad & \frac{(1+x)^2}{(1-x)^3} \\
 &= (1+x)^2 (1-x)^{-3} \\
 &= \left[ (1+2x+x^2) \left( 1+3x+\frac{(-3)(-3-1)}{2!}(-x)^2 + \dots \right) \right] \\
 &= (1+2x+x^2) \left( 1+3x+\frac{-3x-4}{2 \times 1}x^2 + \dots \right) \\
 &= (1+2x+x^2) \\
 &\quad \left( 1+3x+\frac{3 \times 4}{2}x^2+\frac{4 \times 5}{2}x^3+\dots+\frac{(n-1)(n)}{2}x^{n-2}+\frac{n(n+1)}{2}x^{n-1}+\frac{(n+1)(n+2)}{2}x^n \right) \\
 \Rightarrow &= \frac{(n+1)(n+2)}{2}x^n + \frac{2(n)(n+1)}{2}x^{n-1+1} + \frac{(n-1)(n)}{2}x^{n-2+2} \\
 &= \left( \frac{n^2+3n+2+2n^2+2n+n^2-n}{2} \right) x^n \\
 &= \left( \frac{4n^2+4n+2}{2} \right) x^n = (2n^2+2n+1)x^n
 \end{aligned}$$

Hence coefficient of  $x^n$  is  $2n^2+2n+1$ .

$$\text{(v)} \quad (1-x+x^2-x^3+\dots)$$

We know that,

$$(1+x)^{-1} = 1-x+x^2-x^3+\dots$$

Hence given expression becomes

$$\begin{aligned}
 [(1+x)^{-1}]^2 &= (1+x)^{-2} \\
 &= 1+(-2)x+\frac{(-2)(-2-1)}{2!}(-x)^2+\frac{(-2)(-2-1)(-2-2)}{3!}(-x)^3+\dots \\
 &= 1+(-1)2x+\frac{(-2)(-3)}{2 \times 1}x^2+\frac{(-2)(-3)(-4)}{3 \times 2 \times 1}(-x)^3+\dots \\
 &= 1+(-1)2x+(-1)^23x^2+(-1)^34x^3+\dots+(-1)^{n-2}(n-2)x^{n-2} \\
 &\quad +(-1)^{n-1}nx^{n-1}+(-1)^n(n+1)x^n
 \end{aligned}$$

Hence coefficient of  $x^n$  is only,  $(-1)^n(n+1)$



**Q.4** If  $x$  is so small that its square and higher power can be neglected, then show that

(i)  $\frac{1-x}{\sqrt{1-x}} \approx 1 - \frac{3}{2}x$  (Lahore Board 2009)

(ii)  $\frac{\sqrt{1+2x}}{\sqrt{1-x}} \approx 1 + \frac{3}{2}x$

(iii)  $\frac{(9+7x)^{1/2} - (16+3x)^{1/4}}{4+5x} \approx \frac{1}{4} - \frac{17}{284}x$

(iv)  $\frac{\sqrt{4+x}}{(1-x)^3} \approx 2 + \frac{25}{4}x$

(v)  $\frac{(1+x)^{1/2} (4-3x)^{3/2}}{(8+5x)^{1/3}} \approx \left(1 - \frac{5}{6}x\right)$  (Gujranwala Board 2006)

(vi)  $\frac{(1-x)^{1/2} (9-4x)^{1/2}}{(8+3x)^{1/3}} \approx \frac{3}{2} - \frac{61}{48}x$

(vii)  $\frac{\sqrt{4-x} + (8-x)^{1/3}}{(8-x)^{1/3}} \approx 2 - \frac{1}{12}x$

**Solution:**

(i)  $\frac{1-x}{\sqrt{1-x}} \approx 1 - \frac{3}{2}x$

$$\begin{aligned} \text{L.H.S.} &= (1-x)(1+x)^{-1/2} \\ &= (1-x)\left(1 - \frac{1}{2}x\right) \quad (\text{neglecting square and higher power of } x) \\ &= 1 - \frac{1}{2}x - x \\ &= 1 - \frac{3}{2}x \\ &= \text{R.H.S.} \end{aligned}$$

(ii)  $\frac{\sqrt{1+2x}}{\sqrt{1-x}} \approx 1 + \frac{3}{2}x$

$$\begin{aligned} \text{L.H.S.} &= (1+2x)^{1/2} (1-x)^{-1/2} \\ &= \left(1 + \frac{1}{2}2x\right)\left(1 + \frac{1}{2}x\right) \end{aligned}$$



$$= (1+x) \left( 1 + \frac{1}{2}x \right)$$

$$= 1 + \frac{1}{2}x + x$$

$$= 1 + \frac{3}{2}x = \text{R.H.S.}$$

$$(iii) \quad \frac{(9+7x)^{1/2} - (16+3x)^{1/4}}{4+5x} \approx \frac{1}{4} - \frac{17}{284}x$$

$$\begin{aligned} \text{L.H.S.} &= [(9+7x)^{1/2} - (16+3x)^{1/4}] (4+5x)^{-1} \\ &= \left[ 9^{1/2} \left( 1 + \frac{7}{9}x \right)^{1/2} - 16^{1/4} \left( 1 + \frac{3x}{16} \right)^{1/4} \right] \cdot 4^{-1} \left( 1 + \frac{5x}{4} \right)^{-1} \\ &= \left[ 3 \left( 1 + \frac{7}{8}x \right) - 2 \left( 1 + \frac{3}{64}x \right) \right] \frac{1}{4} \left( 1 - \frac{5x}{4} \right) \\ &= \frac{1}{4} \left[ 3 + \frac{7}{6}x - 2 - \frac{3}{32}x \right] \left( 1 - \frac{5}{4}x \right) \\ &= \frac{1}{4} \left[ \left( 1 + \frac{103}{96}x \right) \left( 1 - \frac{5}{4}x \right) \right] \\ &= \frac{1}{4} \left( 1 - \frac{5}{4}x + \frac{103}{96}x \right) \quad [\because \text{neglecting higher power of } x] \\ &= \frac{1}{4} \left( 1 - \frac{17}{96}x \right) \\ &= \frac{1}{4} - \frac{17}{384}x = \text{R.H.S.} \end{aligned}$$

$$(iv) \quad \frac{\sqrt{4+x}}{(1-x)^3} \approx 2 + \frac{25}{4}x$$

$$\begin{aligned} \text{L.H.S.} &= (4+x)^{1/2} (1-x)^{-3} \\ &= 4^{1/2} \left( 1 + \frac{x}{4} \right)^{1/2} (1-x)^{-3} \\ &= 2 \left( 1 + \frac{1}{8}x \right) (1+3x) \\ &= 2 \left( 1 + 3x + \frac{1}{8}x \right) \\ &= 2 \left( 1 + \frac{25}{8}x \right) \\ &= 2 + \frac{25}{4}x = \text{R.H.S.} \end{aligned}$$



$$(v) \quad \frac{(1+x)^{1/2} (4-3x)^{3/2}}{(8+5x)^{1/3}} \approx \left(1 - \frac{5}{6}x\right)$$

$$\begin{aligned} \text{L.H.S.} &= (1+x)^{1/2} (4-3x)^{3/2} (8+5x)^{-1/3} \\ &= (1+x)^{1/2} 4^{3/2} \left(1 - \frac{3x}{4}\right)^{3/2} (8)^{-1/3} \left(1 + \frac{5x}{8}\right)^{-1/3} \\ &= \left(1 + \frac{1}{2}x\right) 2^3 \left(1 - \frac{9}{8}x\right) 2^{-1} \left(1 - \frac{5}{24}x\right) \\ &= 2^3 2^{-1} \left(1 + \frac{1}{2}x\right) \left(1 - \frac{9}{8}x\right) \left(1 - \frac{5}{24}x\right) \\ &= 2^2 \left(1 + \frac{1}{2}x\right) \left(1 - \frac{5}{24}x - \frac{9}{8}x\right) \\ &= 4 \left(1 - \frac{5}{24}x - \frac{9}{8}x + \frac{1}{2}x\right) \\ &= 4 \left(1 - \frac{5}{6}x\right) \\ &= \text{R.H.S.} \end{aligned}$$

$$(vi) \quad \frac{(1-x)^{1/2} (9-4x)^{1/2}}{(8+3x)^{1/3}} \approx \frac{3}{2} - \frac{61}{48}x$$

$$\begin{aligned} \text{L.H.S.} &= (1-x)^{1/2} (9-4x)^{1/2} (8+3x)^{-1/3} \\ &= (1-x)^{1/2} 9^{1/2} \left(1 - \frac{4x}{9}\right)^{1/2} 8^{-1/3} \left(1 + \frac{3x}{8}\right)^{-1/3} \\ &= \left(1 - \frac{1}{2}x\right) 3 \left(1 - \frac{4}{18}x\right) 2^{-1} \left(1 - \frac{3x}{24}\right) \\ &= 3^1 2^{-1} \left(1 - \frac{1}{2}x\right) \left(1 - \frac{2}{9}x\right) \left(1 - \frac{1}{8}x\right) \\ &= \frac{3}{2} \left(1 - \frac{1}{2}x\right) \left(1 - \frac{1}{8}x - \frac{2}{9}x\right) \\ &= \frac{3}{2} \left(1 - \frac{1}{8}x - \frac{2}{9}x - \frac{1}{2}x\right) \\ &= \frac{3}{2} \left(1 - \frac{61}{72}x\right) \end{aligned}$$



$$= \frac{3}{2} - \frac{3}{2} \times \frac{61}{72} x$$

$$= \frac{3}{2} - \frac{61}{48}$$

$$= \text{R.H.S.}$$

$$(vii) \quad \frac{\sqrt{4-x} + (8-x)^{1/3}}{(8-x)^{1/3}} \approx 2 - \frac{1}{12} x$$

$$\begin{aligned} \text{L.H.S.} &= [(4-x)^{1/2} + (8-x)^{1/3}] (8-x)^{-1/3} \\ &= \left[ 4^{1/2} \left(1 - \frac{x}{4}\right)^{1/2} + 8^{1/3} \left(1 - \frac{x}{8}\right)^{1/3} \right] (8)^{-1/3} \left(1 - \frac{x}{8}\right)^{-1/3} \\ &= \left[ 2 \left(1 - \frac{1}{8} x\right) + 2 \left(1 - \frac{x}{24}\right) \right] 2^{-1} \left(1 + \frac{1}{24} x\right) \\ &= \left[ 2 - \frac{1}{4} x + 2 - \frac{x}{12} \right] \frac{1}{2} \left(1 + \frac{1}{24} x\right) \\ &= \frac{1}{2} \left(4 - \frac{1}{3} x\right) \left(1 + \frac{1}{24} x\right) \\ &= \frac{1}{2} \left(4 + \frac{1}{6} x - \frac{1}{3} x\right) \\ &= \frac{1}{2} \left(4 - \frac{1}{6} x\right) \\ &= 2 - \frac{1}{12} x \\ &= \text{R.H.S.} \end{aligned}$$

**Q.5** If  $x$  is so small that its cube and heigher power can be neglected, show that

$$(i) \quad \sqrt{1-x-2x^2} \approx 1 - \frac{1}{2} x - \frac{9}{8} x^2$$

$$(ii) \quad \sqrt{\frac{1+x}{1-x}} \approx 1 + x + \frac{1}{2} x$$

**Solution:**

$$(i) \quad \sqrt{1-x-2x^2} \approx 1 - \frac{1}{2} x - \frac{9}{8} x^2$$

$$\begin{aligned} \text{L.H.S.} &= (1-x-2x^2)^{1/2} \\ &= [1-(x+2x^2)]^{1/2} \end{aligned}$$



$$\begin{aligned}
&= 1 - \frac{1}{2}(x + 2x^2) + \frac{1}{2}\left(\frac{1}{2} - 1\right)[-(x + 2x^2)^2] \\
&= 1 - \frac{1}{2}(x + 2x^2) + \frac{1}{2}\left(-\frac{1}{2}\right) \times \frac{1}{2}(x^2 + 4x^4 + 4x^3) \\
&= 1 - \frac{1}{2}(x + 2x^2) - \frac{1}{8}(x^2 + 4x^4 + 4x^3) \\
&= 1 - \frac{1}{2}x - x^2 - \frac{1}{8}x^2 \quad (\text{neglecting cube and higher power of } x) \\
&= 1 - \frac{1}{2}x - \frac{9}{8}x^2 \\
&= \text{R.H.S.}
\end{aligned}$$

(ii)  $\sqrt{\frac{1+x}{1-x}} \approx 1 + x + \frac{1}{2}x$

$$\begin{aligned}
\text{L.H.S.} &= (1+x)^{1/2} (1-x)^{-1/2} \\
&= \left[1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2\right] \left[1 + \frac{1}{2}x + \frac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)}{2!}(-x)^2\right] \\
&= \left[1 + \frac{1}{2}x - \frac{1}{8}x^2\right] \left[1 + \frac{1}{2}x + \frac{3}{8}x^2\right] \\
&= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{3}{16}x^3 - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{3}{64}x^4 \\
&= 1 + \frac{1}{2}x + \frac{1}{2}x + \frac{3}{8}x^2 - \frac{1}{8}x^2 + \frac{1}{4}x^2 \\
&= 1 + x + \frac{1}{2}x^2 \\
&= \text{R.H.S.}
\end{aligned}$$

**Q.6** If  $x$  is very nearly equal to 1, then prove that

$$Px^p - qx^q \approx (p - q)x^{p+q}$$

(Gujranwala Board 2005, 2003), (Lahore Board 2003, 2009, 2011)

**Solution:**

Since  $x \approx 1$

Let  $x = 1 + h$  where  $h$  is so small that its square and higher powers can be neglected.

$$\begin{aligned}
\text{L.H.S.} &= Px^p - qx^q \\
&\approx P(1+h)^p - q(1+h)^q
\end{aligned}$$



$$\begin{aligned}
&\approx P(1 + ph) - q(1 + qh) \\
&\approx P + p^2 h - q - q^2 h \\
&\approx (p - q) + (p^2 - q^2) h \\
&\approx (p - q) + (p - q)(p + q) h \\
&\approx (p - q)[1 + (p + q) h] \quad \dots\dots\dots (1)
\end{aligned}$$

$$\begin{aligned}
\text{R.H.S.} &= (P - q) x^{p+q} \\
&\approx (P - q)(1 + h)^{p+q} \\
&\approx (P - q)[1 + (p + q) h] \quad \dots\dots\dots (2)
\end{aligned}$$

From (1) and (2) we have

$$\text{L.H.S.} = \text{R.H.S.}$$

Hence proved.

**Q.7 If  $p - q$  is small, when compared with  $p$  or  $q$  show that**

$$\frac{(2n+1)p + (2n-1)q}{(2n-1)p + (2n+1)q} = \left[ \frac{p+q}{2q} \right]^{1/n}$$

**Solution:**

$$\text{L.H.S.} = \frac{(2n+1)p + (2n-1)q}{(2n-1)p + (2n+1)q}$$

$$\text{Let } p - q = h$$

$p = q + h$ , where 'h' is a small that its square and higher powers can be neglected.

$$\begin{aligned}
&= \frac{(2n+1)(q+h) + (2n-1)q}{(2n-1)(q+h) + (2n+1)q} \\
&= \frac{2nq + 2nh + q + h + 2nq - q}{2nq + 2nh - q - h + 2nq + q} \\
&= \frac{4nq + 2nh + h}{4nq + 2nh - h} = \frac{4nq + (2n+1)h}{4nq + (2n-1)h} \\
&= \frac{4nq \left[ 1 + \left( \frac{2n+1}{4nq} \right) h \right]}{4nq \left[ 1 + \left( \frac{2n-1}{4nq} \right) h \right]} \\
&= \left[ 1 + \left( \frac{2n+1}{4nq} \right) h \right] \left[ 1 + \left( \frac{2n-1}{4nq} \right) h \right]^{-1} \\
&= \left[ 1 + \left( \frac{2n+1}{4nq} \right) h \right] \left[ 1 - \left( \frac{2n-1}{4nq} \right) h \right]
\end{aligned}$$



$$\begin{aligned}
 &= 1 + \left( \frac{2n+1}{4nq} \right)h - \left( \frac{2n-1}{4nq} \right)h \\
 &= 1 + \frac{2nh + h - 2nh + h}{4nq} \\
 &= 1 + \frac{2h}{4nq} = 1 + \frac{h}{2nq} \quad \dots\dots\dots (1)
 \end{aligned}$$

$$\begin{aligned}
 \text{R.H.S.} &= \left[ \frac{p+q}{2q} \right]^{1/n} \\
 &= \left[ \frac{q+h+q}{2q} \right]^{1/n} \\
 &= \left[ \frac{2q+h}{2q} \right]^{1/n} = \left[ \frac{2q}{2q} + \frac{h}{2q} \right]^{1/n} \\
 &= \left[ 1 + \frac{h}{2q} \right]^{1/n} = 1 + \frac{h}{2nq} \quad \dots\dots\dots (2)
 \end{aligned}$$

By (1) and (2)

$$\text{L.H.S.} = \text{R.H.S.}$$

**Q.8** Show that  $\left[ \frac{n}{2(n+N)} \right]^{1/2} = \frac{8n}{9n-N} - \frac{n+N}{4n}$  where  $n$  and  $N$  are nearly equal.

**Solution:**

Since,  $N \approx n$

$\Rightarrow N = n + h$ , where 'h' is so small that its square and higher powers can be neglected.

$$\begin{aligned}
 \text{L.H.S.} &= \left[ \frac{n}{2(n+N)} \right]^{1/2} \\
 &= \left[ \frac{n}{2(n+n+h)} \right]^{1/2} = \left[ \frac{n}{2(2n+h)} \right]^{1/2} = \left[ \frac{n}{4n+2h} \right]^{1/2} \\
 &= \left[ \frac{n}{4n \left( 1 + \frac{2h}{4n} \right)} \right]^{1/2} = \left[ \frac{1}{4^{1/2} \left( 1 + \frac{2h}{4n} \right)^{1/2}} \right] \\
 &= \frac{1}{\sqrt{4}} \left[ 1 + \frac{2h}{4n} \right]^{-1/2} = \frac{1}{2} \left[ 1 - \frac{2h}{8n} \right] \\
 &= \frac{1}{2} \left[ 1 - \frac{h}{4n} \right] \quad \dots\dots\dots (1)
 \end{aligned}$$



$$\begin{aligned}
 \text{R.H.S.} &= \frac{8n}{9n-n} - \frac{n+N}{4n} \\
 &= \frac{8n}{9n-(n+h)} - \frac{(n+n+h)}{4n} \\
 &= \frac{8n}{9n-n-h} - \frac{2n+h}{4n} \\
 &= \frac{8n}{8n-h} - \frac{2n+h}{4n} \\
 &= \frac{8n}{8n\left(1-\frac{h}{8n}\right)} - \frac{2n+h}{4n} \\
 &= \left(1-\frac{h}{8n}\right)^{-1} - \frac{2n+h}{4n} \\
 &\approx 1 + \frac{h}{8n} - \left(\frac{2n+h}{4n}\right) \\
 &\approx \frac{8n+h-4n-2h}{8n} \\
 &\approx \frac{4n-h}{8n} \approx \frac{4n}{8n} - \frac{h}{8n} \approx \frac{1}{2} - \frac{h}{8n} \\
 &\approx \frac{1}{2} \left[1 - \frac{h}{4n}\right] \quad \dots\dots\dots (2)
 \end{aligned}$$

From (1) and (2), we have

$$\text{L.H.S.} = \text{R.H.S.}$$

**Q.9 Identify the following series as binomial expansion and find the sum in ease.**

(i)  $1 - \frac{1}{2}\left(\frac{1}{4}\right) + \frac{1 \cdot 3}{2! \cdot 4}\left(\frac{1}{4}\right)^2 - \frac{1 \cdot 3 \cdot 5}{3! \cdot 8}\left(\frac{1}{4}\right)^3 + \dots\dots$

(ii)  $1 - \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1 \cdot 3}{2 \cdot 4}\left(\frac{1}{2}\right)^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\left(\frac{1}{2}\right)^3 + \dots\dots$

(iii)  $1 + \frac{3}{4} + \frac{3 \cdot 5}{4 \cdot 8} + \frac{3 \cdot 5 \cdot 7}{4 \cdot 8 \cdot 12} + \dots\dots$

(iv)  $1 - \frac{1}{2} \cdot \frac{1}{3} + \frac{1 \cdot 3}{2 \cdot 4}\left(\frac{1}{3}\right)^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\left(\frac{1}{3}\right)^3 + \dots\dots$



**Solution:**

$$(i) \quad 1 - \frac{1}{2} \left( \frac{1}{4} \right) + \frac{1 \cdot 3}{2! \cdot 4} \left( \frac{1}{4} \right)^2 - \frac{1 \cdot 3 \cdot 5}{3! \cdot 8} \left( \frac{1}{4} \right)^3 + \dots$$

$$\text{Let } (1+x)^n = 1 - \frac{1}{2} \left( \frac{1}{4} \right) + \frac{1 \cdot 3}{2! \cdot 4} \left( \frac{1}{4} \right)^2 - \dots$$

$$\text{Also, } (1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \dots$$

Now, comparing term by term of the above two equations, we have

$$nx = \frac{-1}{8} \quad \dots \quad (1)$$

$$\frac{n(n-1)}{2!} x^2 = \frac{3}{128} \quad \dots \quad (2)$$

$$\therefore \frac{1 \cdot 3}{2! \cdot 4} \left( \frac{1}{4} \right)^2 = \frac{1 \cdot 3}{2 \cdot 1 \cdot 4} \frac{1}{16} = \frac{3}{128}$$

By (1), we have

$$x = \frac{-1}{8n} \quad \dots \quad (3)$$

Putting the value of  $x$  in (2)

$$\frac{n(n-1)}{2!} \cdot \frac{1}{64 n^2} = \frac{3}{128}$$

$$\frac{n-1}{128n} = \frac{3}{128}$$

Multiplying both sides by 128

$$\frac{n-1}{n} = 3 \Rightarrow n-1 = 3n$$

$$\Rightarrow -1 = 2n$$

$$\Rightarrow \boxed{n = \frac{-1}{2}}$$

Putting value of  $n$  in (3)

$$x = -\frac{1}{48 \left( \frac{-1}{2} \right)}$$

$$\Rightarrow \boxed{x = \frac{1}{4}}$$



Now, putting the values of  $x$  and  $n$  in,

$$(1+x)^n = \left(1 + \frac{1}{4}\right)^{-1/2} = \left(\frac{5}{4}\right)^{-1/2}$$

$$\text{Required sum} = \left(\frac{4}{5}\right)^{1/2} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}$$

$$(ii) \quad 1 - \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1 \cdot 3}{2 \cdot 4}\left(\frac{1}{2}\right)^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\left(\frac{1}{2}\right)^3 + \dots$$

$$\text{Let } (1+x)^n = 1 - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \frac{1 \cdot 3}{2 \cdot 4}\left(\frac{1}{2}\right)^2 - \dots$$

$$\text{Also, } (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots$$

Now comparing term by term of the above two equations, we have

$$nx = -\frac{1}{4} \quad \dots\dots\dots (1)$$

$$\frac{n(n-1)}{2!}x^2 = \frac{3}{32} \quad \dots\dots\dots (2)$$

$$\therefore \frac{1 \cdot 3}{2 \cdot 4} \frac{1}{4} = \frac{3}{32}$$

By (1), we have

$$x = -\frac{1}{4n} \quad \dots\dots\dots (3)$$

[Putting the value of  $x$  in (2)]

$$\frac{n(n-1)}{2!} \frac{1}{16n^2} = \frac{3}{32}$$

$$\frac{n-1}{32n} = \frac{3}{32}$$

$$\Rightarrow \frac{n-1}{n} = 3 \quad (\because \text{since multiply both sides by } 32)$$

$$\Rightarrow n-1 = 3n$$

$$\Rightarrow -1 = 3n - n$$

$$-1 = 2n$$

$$\Rightarrow \boxed{n = -\frac{1}{2}}$$



Putting the value of  $n$  in (3)

$$x = -\frac{1}{2 + \left(-\frac{1}{2}\right)}$$

$$\boxed{x = \frac{1}{2}}$$

Now, putting the values of  $x$  and  $n$  in

$$(1+x)^n = \left(1 + \frac{1}{2}\right)^{-1/2} = \left(\frac{3}{2}\right)^{-1/2} = \left(\frac{2}{3}\right)^{1/2} = \sqrt{\frac{2}{3}}, \text{ required sum}$$

$$(iii) \quad 1 + \frac{3}{4} + \frac{3 \cdot 5}{4 \cdot 8} + \frac{3 \cdot 5 \cdot 7}{4 \cdot 8 \cdot 12} + \dots$$

$$\text{Let } (1+x)^n = 1 + \frac{3}{4} + \frac{3 \cdot 5}{4 \cdot 8} + \dots$$

$$\text{Also, } (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots$$

Now comparing term by term above two equations, we have

$$nx = \frac{3}{4} \quad \dots\dots\dots (1)$$

$$\frac{n(n-1)}{2!}x^2 = \frac{15}{32} \quad \dots\dots\dots (2)$$

By (1), we have

$$x = \frac{3}{4n} \quad \dots\dots\dots (3)$$

Putting the value of  $x$  in (2)

$$\frac{n(n-1)}{2!} \frac{9}{16n^2} = \frac{15}{32}$$

$$\frac{(n-1)9}{32n} = \frac{15}{32}$$

$$\frac{(n-1)9}{n} = 15 \quad (\because \text{multiplying both sides by } 32)$$

$$9n - 9 = 15n$$

$$-9 = 15n - 9n$$



$$-9 = 6n \Rightarrow n = \frac{-9}{6} \Rightarrow \boxed{n = \frac{-3}{2}}$$

Putting the value of  $n$  in (3).

$$x = \frac{3}{4\left(\frac{-3}{2}\right)}$$

$$\Rightarrow \boxed{x = \frac{-1}{2}}$$

Now putting the values of  $x$  and  $n$  in,

$$(1+x)^n = \left(1 - \frac{1}{2}\right)^{-3/2} = \left(\frac{1}{2}\right)^{-3/2} = (2)^{3/2} = 2 \cdot \sqrt{2}$$

$$(iv) \quad 1 - \frac{1}{2} \cdot \frac{1}{3} + \frac{1 \cdot 3}{2 \cdot 4} \left(\frac{1}{3}\right)^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \left(\frac{1}{3}\right)^3 + \dots$$

$$\text{Let } (1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \dots$$

Now comparing term by term of the above two equations, we have

$$nx = \frac{-1}{6} \quad \dots\dots\dots (1)$$

$$\frac{n(n-1)}{2!} x^2 = \frac{3}{8} \cdot \frac{1}{9}$$

$$\frac{n(n-1)}{2!} x^2 = \frac{1}{24} \quad \dots\dots\dots (2)$$

From (1)  $x = \frac{-1}{6n}$  Put in

$$\frac{n^2 - n}{2} \left(\frac{1}{36n^2}\right) = \frac{1}{24}$$

$$\frac{n(n-1)}{2} \times \frac{1}{36n^2} = \frac{1}{24}$$

$$\frac{n-1}{72n} = \frac{1}{24}$$

$$24n - 24 = 72n$$

$$-24 = 48n$$



$$\boxed{n = \frac{-1}{2}} \quad \text{Put in (1)}$$

$$x = \frac{-1}{6} \times \frac{-2}{1}$$

$$x = \frac{1}{3}$$

$$\text{Hence } \left(1 + \frac{1}{3}\right)^{-1/2} = \left(\frac{4}{3}\right)^{-1/2} = \left(\frac{3}{4}\right)^{1/2} = \frac{\sqrt{3}}{2}$$

**Q.10 Use binomial theorem to show that,  $1 + \frac{1}{4} + \frac{1.3}{4.8} + \frac{1.3.5}{4.8.12} + \dots = \sqrt{2}$**

**Solution:**

L.H.S.

$$\text{Let } (1+x)^n = 1 + \frac{1}{4} + \frac{1.3}{4.8} + \dots$$

$$\text{Also, } (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots$$

Now comparing term by term the above two equations, we have

$$nx = \frac{1}{4} \quad \dots\dots\dots (1)$$

$$\frac{n(n-1)}{2!}x^2 = \frac{3}{32} \quad \dots\dots\dots (2)$$

By (1), we have

$$x = \frac{1}{4n} \quad \dots\dots\dots (3)$$

Putting the value of  $x$  in (2)

$$\frac{n(n-1)}{2!} \frac{1}{16n^2} = \frac{3}{32}$$

$$\frac{n-1}{32n} = \frac{3}{32}$$

$$\frac{n-1}{n} = 3 \quad (\because \text{multiplying both sides by } 32)$$

$$n-1 = 3n$$

$$-1 = 3n - n$$



$$-1 = 2n \quad \Rightarrow \quad \boxed{n = \frac{-1}{2}}$$

Putting the values of  $n$  and  $x$  in (3)

$$(1+x)^n = \left(1 - \frac{1}{2}\right)^{-1/2} = \left(\frac{1}{2}\right)^{-1/2} = (2)^{1/2} = \sqrt{2} \text{ R.H.S.}$$

Hence proved.

**Q.11** If  $y = \frac{1}{3} + \frac{1.3}{2!} \left(\frac{1}{3}\right)^2 + \frac{1.3.5}{3!} \left(\frac{1}{3}\right)^3 + \dots$  then prove that  $y^2 + 2y - 2 = 0$

**Solution:**

By adding '1' on both sides,

$$1+y = 1 + \frac{1}{3} + \frac{1.3}{2!} \left(\frac{1}{3}\right)^2 + \frac{1.3.5}{3!} \left(\frac{1}{3}\right)^3 + \dots$$

$$\text{Let } (1+x)^n = 1 + \frac{1}{3} + \frac{1.3}{2!} \left(\frac{1}{3}\right)^2 + \dots$$

$$\text{Also, } (1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \dots$$

Now comparing term by term above two equations, we have

$$nx = \frac{1}{3} \quad \dots\dots\dots (1)$$

$$\frac{n(n-1)x^2}{2!} = \frac{1.3}{2!} \left(\frac{1}{3}\right)^2 = \frac{3}{2} \left(\frac{1}{9}\right) = \frac{3}{18} \quad \dots\dots\dots (2)$$

By (1) we have,

$$x = \frac{1}{3n} \quad \dots\dots\dots (3)$$

Putting the value of  $x$  in (2)

$$\frac{n(n-1)}{2} \left(\frac{1}{3n}\right)^2 = \frac{3}{18}$$

$$\frac{n(n-1)}{2} \times \frac{1}{9n^2} = \frac{3}{18}$$

$$\frac{n-1}{18n} = \frac{3}{18}$$

$$\frac{n-1}{n} = 3 \quad (\because \text{multiplying both sides by } 18)$$



$$n - 1 = 3n$$

$$\Rightarrow \boxed{n = \frac{-1}{2}}$$

Putting the value of  $n$  in (3)

$$x = \frac{1}{3\left(\frac{-1}{2}\right)} = \frac{1}{\frac{-3}{2}} = \frac{-2}{3}$$

$$\Rightarrow \boxed{x = \frac{-2}{3}}$$

Now, putting the values of  $x$  and  $n$  in,

$$(1+x)^n = \left(1 + \left(\frac{-2}{3}\right)\right)^{-1/2} = \left(1 - \frac{2}{3}\right)^{-1/2}$$

$$(1+y) = \left(\frac{1}{3}\right)^{-1/2} = \sqrt{3}$$

Taking square on both sides,

$$(1+y)^2 = (\sqrt{3})^2$$

$$1 + y^2 + 2y = 3$$

$$y^2 + 2y + 1 = 3$$

$$\Rightarrow y^2 + 2y + 1 - 3 = 0$$

$$\Rightarrow y^2 + 2y - 2 = 0$$

Hence proved.

**Q.12** If  $2y = \frac{1}{2^2} + \frac{1.3}{2!} \cdot \frac{1}{2^4} + \frac{1.3.5}{3!} \left(\frac{1}{2^6}\right) + \dots$  then prove that  $4y^2 + 4y - 1 = 0$ .

(Lahore Board 2006)

**Solution:**

By adding '1' on both sides

$$1 + 2y = 1 + \frac{1}{2^2} + \frac{1.3}{2!} \frac{1}{2^4} + \frac{1.3.5}{3!} \frac{1}{2^6} + \dots$$

$$\text{Let } (1+x)^n = 1 + \frac{1}{2^2} + \frac{1.3}{2!} \frac{1}{2^4} + \dots$$

$$\text{Also, } (1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \dots$$



Now, comparing term by term of the above two equations, we have

$$nx = \frac{1}{2^2} = \frac{1}{4} \quad \dots\dots\dots (1)$$

$$\frac{n(n-1)}{2!} x^2 = \frac{1 \cdot 3}{2!} \frac{1}{2^4} = \frac{3}{2} \times \frac{1}{16} = \frac{3}{32} \quad \dots\dots\dots (2)$$

By (1) we have,

$$x = \frac{1}{4n} \quad \dots\dots\dots (3)$$

Putting the value of  $x$  in (2)

$$\frac{n(n-1)}{2!} \left( \frac{1}{4n} \right)^2 = \frac{3}{32}$$

$$\frac{n(n-1)}{2!} \times \frac{1}{16n^2} = \frac{3}{32}$$

$$\frac{n-1}{2} \frac{1}{16n} = \frac{3}{32}$$

$$\frac{n-1}{32n} = \frac{3}{32}$$

$$\frac{n-1}{n} = 3 \quad (\because \text{multiplying both sides by } 32)$$

$$\frac{n-1}{n} = 3$$

$$\Rightarrow n-1 = 3n$$

$$\Rightarrow \boxed{n = -\frac{1}{2}}$$

Putting the value of  $n$  in (3)

$$x = \frac{1}{2^{4(-1/2)}}$$

$$\Rightarrow \boxed{x = \frac{-1}{2}}$$

Now putting the values of  $x$  and  $n$  in

$$(1+x)^n = \left( 1 + \left( \frac{-1}{2} \right) \right)^{-1/2}$$



$$(1 + 2y) = \left(1 - \frac{1}{2}\right)^{-1/2} = \left(\frac{1}{2}\right)^{-1/2}$$

$$(1 + 2y) = \sqrt{2}$$

Taking square on both sides

$$(1 + 2y)^2 = (\sqrt{2})^2$$

$$1 + 4y^2 + 4y = 2$$

$$\Rightarrow 4y^2 + 4y + 1 - 2 = 0$$

$$\Rightarrow 4y^2 + 4y - 1 = 0$$

Hence proved.

**Q.13** If  $y = \frac{2}{5} + \frac{1.3}{2!} \left(\frac{2}{5}\right)^2 + \frac{1.3.5}{3!} \left(\frac{2}{5}\right)^3 + \dots$  then prove that  $y^2 + 2y - 4 = 0$ .

(Gujranwala Board 2003)

**Solution:**

By adding '1' on both sides,

$$1 + y = 1 + \frac{2}{5} + \frac{1.3}{2!} \left(\frac{2}{5}\right)^2 + \frac{1.3.5}{3!} \left(\frac{2}{5}\right)^3 + \dots$$

$$\text{Let } (1 + x)^n = 1 + \frac{2}{5} + \frac{1.3}{2!} \left(\frac{2}{5}\right)^2 + \dots$$

$$\text{Also, } (1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \dots$$

Now, comparing term by term of the above two equations we have

$$nx = \frac{2}{5} \quad \dots \dots \dots (1)$$

$$\frac{n(n-1)}{2!} x^2 = \frac{1.3}{2!} \left(\frac{2}{5}\right)^2 = \frac{3}{2} \times \frac{4}{25} = \frac{6}{25} \quad \dots \dots \dots (2)$$

By (1) we have,

$$x = \frac{2}{5n} \quad \dots \dots \dots (3)$$

Putting the value of  $x$  in (2)

$$\frac{n(n-1)}{2} \times \left(\frac{2}{5n}\right)^2 = \frac{6}{25}$$



$$\frac{n(n-1)}{2} \times \frac{4}{25n^2} = \frac{6}{25}$$

$$\frac{n-1}{2} \times \frac{4}{25} = \frac{6}{25}$$

$$\frac{2(n-1)}{25n} = \frac{6}{25}$$

$$\frac{2(n-1)}{n} = 6 \quad (\because \text{multiplying both sides by } 25)$$

$$2(n-1) = 6n$$

$$n-1 = 3n \Rightarrow \boxed{n = \frac{-1}{2}}$$

Putting the value of  $n$  in (3)

$$x = \frac{2}{5\left(\frac{-1}{2}\right)} = \frac{2}{-\frac{5}{2}}$$

$$\boxed{x = \frac{-4}{5}}$$

Now, putting the values of  $x$  and  $n$  in,

$$(1+x)^n = \left(1 - \frac{4}{5}\right)^{-1/2}$$

$$(1+y) = \left(\frac{1}{5}\right)^{-1/2}$$

$$(1+y) = \sqrt{5}$$

Taking square on both sides,

$$(1+y)^2 = (\sqrt{5})^2$$

$$1 + y^2 + 2y = 5$$

$$\Rightarrow y^2 + 2y - 4 = 0$$

Hence proved.

