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CHAPTER

2

DIFFERENTIATION

EXERCISE 2.1

Q.1 Find by definition, the derivatives w.r.t 'x' of the following functions defined as:

- (i) $2x^2 + 1$ (ii) $2 - \sqrt{x}$ (iii) $\frac{1}{\sqrt{x}}$ (iv) $\frac{1}{x^3}$
 (v) $\frac{1}{x-a}$ (vi) $x(x-3)$ (vii) $\frac{2}{x^4}$ (viii) $x^2 + \frac{1}{x^2}$
 (ix) $(x+4)^{\frac{1}{3}}$ (x) $x^{\frac{3}{2}}$ (xi) $x^{\frac{5}{2}}$ (xii) x^m
 (xiii) $\frac{1}{x^m}$, $m \in \mathbb{N}$ (xiv) x^{40} (xv) x^{-100}

Solution:

(i) $2x^2 + 1$ (Lahore Board 2011)

Let

$$y = 2x^2 + 1$$

$$y + \delta y = 2(x + \delta x)^2 + 1$$

$$\delta y = 2(x + \delta x)^2 + 1 - y$$

$$\delta y = 2(x^2 + \delta x^2 + 2x\delta x) + 1 - (2x^2 + 1) \quad \because y = 2x^2 + 1$$

$$\delta y = 2x^2 + 2\delta x^2 + 4x\delta x + 1 - 2x^2 - 1$$

$$\delta y = 2\delta x^2 + 4x\delta x$$

$$\delta y = 2\delta x (\delta x + 2x)$$

Dividing both sides by δx .

$$\frac{\delta y}{\delta x} = \frac{2\delta x(\delta x + 2x)}{\delta x}$$

$$\frac{\delta y}{\delta x} = 2(\delta x + 2x)$$

Taking limit $\delta x \rightarrow 0$



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$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} [2(\delta x + 2x)]$$

$$\frac{dy}{dx} = 2(0 + 2x)$$

$$\frac{dy}{dx} = 4x$$

$$\boxed{\frac{d}{dx} (2x^2 + 1) = 4x} \quad \text{Ans.}$$

(ii) $2 - \sqrt{x}$

Let

$$y = 2 - \sqrt{x}$$

$$y + \delta y = 2 - \sqrt{x + \delta x}$$

$$\delta y = 2 - \sqrt{x + \delta x} - y$$

$$\delta y = 2 - \sqrt{x + \delta x} - (2 - \sqrt{x}) \quad \because y = 2 - \sqrt{x}$$

$$\delta y = 2 - \sqrt{x + \delta x} - 2 + \sqrt{x}$$

$$\delta y = \sqrt{x} - \sqrt{x + \delta x}$$

$$\delta y = x^{\frac{1}{2}} - \left[x \left(1 + \frac{\delta x}{x} \right) \right]^{\frac{1}{2}}$$

$$\delta y = x^{\frac{1}{2}} - x^{\frac{1}{2}} \left(1 + \frac{\delta x}{x} \right)^{\frac{1}{2}}$$

$$\delta y = x^{\frac{1}{2}} \left[1 - \left(1 + \frac{\delta x}{x} \right)^{\frac{1}{2}} \right]$$

$$\delta y = x^{\frac{1}{2}} \left[1 - \left\{ 1 + \frac{1}{2} \left(\frac{\delta x}{x} \right) + \frac{\frac{1}{2}(\frac{1}{2} - 1)}{2!} \left(\frac{\delta x}{x} \right)^2 + \dots \right\} \right]$$

$$\delta y = x^{\frac{1}{2}} \left[1 - 1 - \frac{1}{2} \left(\frac{\delta x}{x} \right) - \frac{\frac{1}{2}(\frac{1}{2} - 1)}{2!} \cdot \left(\frac{\delta x}{x} \right)^2 - \dots \right]$$

$$\delta y = x^{\frac{1}{2}} \left[-\frac{1}{2} \left(\frac{\delta x}{x} \right) - \frac{\frac{1}{2}(\frac{1}{2} - 1)}{2!} \cdot \left(\frac{\delta x}{x} \right)^2 - \dots \right]$$



$$\delta y = x^{\frac{1}{2}} \frac{\delta x}{x} \left[\frac{-1}{2} - \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!} \cdot \frac{\delta x}{x} - \dots \right]$$

Dividing both sides by δx .

$$\frac{\delta y}{\delta x} = \frac{\delta x}{\delta x} x^{\frac{1}{2}-1} \left[\frac{-1}{2} - \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!} \cdot \frac{\delta x}{x} - \dots \right]$$

$$\frac{\delta y}{\delta x} = \frac{1}{x^{\frac{1}{2}}} \left[\frac{-1}{2} - \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!} \cdot \frac{\delta x}{x} - \dots \right]$$

Taking limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{1}{x^{\frac{1}{2}}} \left[\frac{-1}{2} - \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!} \cdot \frac{\delta x}{x} - \dots \right]$$

$$\frac{dy}{dx} = \frac{-1}{2\sqrt{x}}$$

$$\boxed{\frac{dy}{dx} (2 - \sqrt{x}) = \frac{-1}{2\sqrt{x}}} \quad \text{Ans.}$$

(iii) $\frac{1}{\sqrt{x}}$ (L.B 2007)

Let

$$y = \frac{1}{\sqrt{x}}$$

$$y = x^{-1/2}$$

$$y + \delta y = (x + \delta x)^{-1/2}$$

$$\delta y = (x + \delta x)^{-1/2} - y$$

$$\delta y = (x + \delta x)^{-1/2} - x^{-1/2} \quad \because y = x^{-1/2}$$

$$\delta y = \left[x \left(1 + \frac{\delta x}{x} \right) \right]^{-1/2} - x^{-1/2}$$

$$\delta y = x^{-1/2} \left(1 + \frac{\delta x}{x} \right)^{-1/2} - x^{-1/2}$$

$$\delta y = x^{-1/2} \left[\left(1 + \frac{\delta x}{x} \right)^{-1/2} - 1 \right]$$



$$\delta y = x^{-1/2} \left[1 + \left(\frac{-1}{2} \right) \left(\frac{\delta x}{x} \right) + \frac{-1/2 (-1/2 - 1)}{2!} \left(\frac{\delta x}{x} \right)^2 + \dots - 1 \right]$$

$$\delta y = x^{-1/2} \frac{\delta x}{x} \left[\frac{-1}{2} + \frac{-1/2 (-1/2 - 1)}{2!} \cdot \frac{\delta x}{x} + \dots \right]$$

Dividing both sides by δx .

$$\frac{\delta y}{\delta x} = \frac{\delta x}{\delta x} x^{-1/2} \left[\frac{-1}{2} + \frac{-1/2 (-1/2 - 1)}{2!} \cdot \frac{\delta x}{x} + \dots \right]$$

$$\frac{\delta y}{\delta x} = \frac{1}{x^{3/2}} \left[\frac{-1}{2} + \frac{-1/2 (-1/2 - 1)}{2!} \cdot \frac{\delta x}{x} + \dots \right]$$

Taking limit $\delta x \rightarrow 0$.

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{1}{x^{3/2}} \left[\frac{-1}{2} + \frac{-1/2 (-1/2 - 1)}{2!} \cdot \frac{\delta x}{x} + \dots \right]$$

$$\frac{dy}{dx} = \frac{-1}{2 x^{3/2}}$$

$$\boxed{\frac{d}{dx} \left(\frac{1}{\sqrt{x}} \right) = \frac{-1}{2 x^{3/2}}}$$

Ans.

(iv) $\frac{1}{x^3}$
Let

$$y = \frac{1}{x^3}$$

$$y = x^{-3}$$

$$y + \delta y = (x + \delta x)^{-3}$$

$$\delta y = (x + \delta x)^{-3} - y$$

$$\delta y = (x + \delta x)^{-3} - x^{-3}$$

$$\because y = x^{-3}$$

$$\delta y = \left[x \left(1 + \frac{\delta x}{x} \right) \right]^{-3} - x^{-3}$$

$$\delta y = x^{-3} \left(1 + \frac{\delta x}{x} \right)^{-3} - x^{-3}$$

$$\delta y = x^{-3} \left[\left(1 + \frac{\delta x}{x} \right)^{-3} - 1 \right]$$

$$\delta y = x^{-3} \left[1 + (-3) \left(\frac{\delta x}{x} \right) + \frac{(-3)(-3-1)}{2!} \cdot \left(\frac{\delta x}{x} \right)^2 + \dots - 1 \right]$$



$$\delta y = x^{-3} \cdot \frac{\delta x}{x} \left[-3 + \frac{(-3)(-3-1)}{2!} \cdot \frac{\delta x}{x} + \dots \right]$$

Dividing both sides by δx .

$$\frac{\delta y}{\delta x} = \frac{\delta x}{\delta x} x^{-3} \left[-3 + \frac{(-3)(-3-1)}{2!} \cdot \frac{\delta x}{x} + \dots \right]$$

$$\frac{\delta y}{\delta x} = \frac{1}{x^4} \left[-3 + \frac{(-3)(-3-1)}{2!} \cdot \frac{\delta x}{x} + \dots \right]$$

Taking limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{1}{x^4} \left[-3 + \frac{(-3)(-3-1)}{2!} \cdot \frac{\delta x}{x} + \dots \right]$$

$$\frac{dy}{dx} = \frac{-3}{x^4}$$

$$\boxed{\frac{d}{dx} \left(\frac{1}{x^3} \right) = \frac{-3}{x^4}} \quad \text{Ans.}$$

(v) $\frac{1}{x-a}$
Let

$$y = \frac{1}{x-a}$$

$$y = (x-a)^{-1}$$

$$y + \delta y = (x + \delta x - a)^{-1}$$

$$\delta y = (x-a+\delta x)^{-1} - y$$

$$\delta y = \left[(x-a) \left(1 + \frac{\delta x}{x-a} \right) \right]^{-1} - (x-a)^{-1} \quad \because y = (x-a)^{-1}$$

$$\delta y = (x-a)^{-1} \left(1 + \frac{\delta x}{x-a} \right)^{-1} - (x-a)^{-1}$$

$$\delta y = (x-a)^{-1} \left[\left(1 + \frac{\delta x}{x-a} \right)^{-1} - 1 \right]$$

$$\delta y = (x-a)^{-1} \left[1 + (-1) \left(\frac{\delta x}{x-a} \right) + \frac{(-1)(-1-1)}{2!} \cdot \left(\frac{\delta x}{x-a} \right)^2 + \dots - 1 \right]$$

$$\delta y = (x-a)^{-1} \cdot \frac{\delta x}{x-a} \left[-1 + \frac{(-1)(-1-1)}{2!} \cdot \frac{\delta x}{x-a} + \dots \right]$$



Dividing both sides by δx

$$\frac{\delta y}{\delta x} = \frac{\delta x}{\delta x (x-a)^{1+1}} \left[-1 + \frac{(-1)(-1-1)}{2!} \cdot \frac{\delta x}{x-a} + \dots \right]$$

$$\frac{\delta y}{\delta x} = \frac{1}{(x-a)^2} \left[-1 + \frac{(-1)(-1-1)}{2!} \cdot \frac{\delta x}{x-a} + \dots \right]$$

Taking limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{1}{(x-a)^2} \left[-1 + \frac{(-1)(-1-1)}{2!} \cdot \frac{\delta x}{x-a} + \dots \right]$$

$$\frac{dy}{dx} = \frac{-1}{(x-a)^2}$$

$$\boxed{\frac{d}{dx} \left(\frac{1}{x-a} \right) = \frac{-1}{(x-a)^2}} \quad \text{Ans.}$$

(vi) $x(x-3)$

Let $y = x(x-3)$

$$y = x^2 - 3x$$

$$y + \delta y = (x + \delta x)^2 - 3(x + \delta x)$$

$$\delta y = x^2 + \delta x^2 + 2x\delta x - 3x - 3\delta x - y$$

$$\delta y = x^2 + \delta x^2 + 2x\delta x - 3x - 3\delta x - x^2 + 3x$$

$$\delta y = \delta x(\delta x + 2x - 3)$$

Dividing both sides by δx .

$$\frac{\delta y}{\delta x} = \frac{\delta x(\delta x + 2x - 3)}{\delta x} = \delta x + 2x - 3$$

Taking limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} (\delta x + 2x - 3)$$

$$\frac{dy}{dx} = 2x - 3$$

$$\boxed{\frac{dy}{dx} [x(x-3)] = 2x - 3} \quad \text{Ans.}$$

(vii) $\frac{2}{x^4}$

Let $y = \frac{2}{x^4}$

$$y = 2x^{-4}$$



$$\begin{aligned}
 y + \delta y &= 2(x + \delta x)^{-4} \\
 \delta y &= 2(x + \delta x)^{-4} - y \\
 \delta y &= 2 \left[x \left(1 + \frac{\delta x}{x} \right) \right]^{-4} - 2x^{-4} \quad \because y = 2x^{-4} \\
 \delta y &= 2x^{-4} \left(1 + \frac{\delta x}{x} \right)^{-4} - 2x^{-4} \\
 \delta y &= 2x^{-4} \left[\left(1 + \frac{\delta x}{x} \right)^{-4} - 1 \right] \\
 \delta y &= 2x^{-4} \left[1 + (-4) \left(\frac{\delta x}{x} \right) + \frac{(-4)(-4-1)}{2!} \left(\frac{\delta x}{x} \right)^2 + \dots - 1 \right] \\
 \delta y &= 2x^{-4} \frac{\delta x}{x} \left[-4 + \frac{(-4)(-4-1)}{2!} \cdot \frac{\delta x}{x} + \dots \right]
 \end{aligned}$$

Dividing both sides by δx .

$$\begin{aligned}
 \frac{\delta y}{\delta x} &= \frac{2\delta x}{\delta x \cdot x^{1+4}} \left[-4 + \frac{(-4)(-4-1)}{2!} \cdot \frac{\delta x}{x} + \dots \right] \\
 \frac{\delta y}{\delta x} &= \frac{2}{x^5} \left[-4 + \frac{(-4)(-4-1)}{2!} \cdot \frac{\delta x}{x} + \dots \right]
 \end{aligned}$$

Taking limit $\delta x \rightarrow 0$

$$\begin{aligned}
 \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} &= \lim_{\delta x \rightarrow 0} \frac{2}{x^5} \left[-4 + \frac{(-4)(-4-1)}{2!} \cdot \frac{\delta x}{x} + \dots \right] \\
 \frac{dy}{dx} &= \frac{2}{x^5} (-4) = \frac{-8}{x^5}
 \end{aligned}$$

$$\boxed{\frac{d}{dx} \left(\frac{2}{x^4} \right) = \frac{-8}{x^5}} \quad \text{Ans.}$$

(viii) $x^2 + \frac{1}{x^2}$

Let $y = x^2 + \frac{1}{x^2}$

$$y = x^2 + x^{-2}$$

$$y + \delta y = (x + \delta x)^2 + (x + \delta x)^{-2}$$

$$\delta y = x^2 + \delta x^2 + 2x\delta x + \left[x \left(1 + \frac{\delta x}{x} \right) - y \right]^{-2} - y$$

$$\delta y = x^2 + \delta x^2 + 2x\delta x + x^{-2} \left(1 + \frac{\delta x}{x} \right)^{-2} - (x^2 + x^{-2}) \quad \because y = x^2 + x^{-2}$$



$$\delta y = x^2 + \delta x^2 + 2x\delta x + x^{-2} \left(1 + \frac{\delta x}{x}\right)^{-2} - x^2 - x^{-2}$$

$$\delta y = \delta x^2 + 2x\delta x + x^{-2} \left[\left(1 + \frac{\delta x}{x}\right)^{-2} - 1 \right]$$

$$\delta y = \delta x(\delta x + 2x) + x^{-2} \left[1 + (-2) \left(\frac{\delta x}{x}\right) + \frac{(-2)(-2-1)}{2!} \cdot \left(\frac{\delta x}{x}\right)^2 + \dots - 1 \right]$$

$$\delta y = \delta x(\delta x + 2x) + x^{-2} \frac{\delta x}{x} \left[-2 + \frac{(-2)(-2-1)}{2!} \cdot \frac{\delta x}{x} + \dots \right]$$

$$\delta y = \delta x \left[\delta x + 2x + \frac{1}{x^{1+2}} \left\{ -2 + \frac{(-2)(-2-1)}{2!} \frac{\delta x}{x} + \dots \right\} \right]$$

Dividing both sides by δx .

$$\frac{\delta y}{\delta x} = \frac{\delta x}{\delta x} \left[\delta x + 2x + \frac{1}{x^3} \left\{ -2 + \frac{(-2)(-2-1)}{2!} \cdot \frac{\delta x}{x} + \dots \right\} \right]$$

$$\frac{\delta y}{\delta x} = \delta x + 2x + \frac{1}{x^3} \left[-2 + \frac{(-2)(-2-1)}{2!} \cdot \frac{\delta x}{x} + \dots \right]$$

Taking limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \left[\delta x + 2x + \frac{1}{x^3} \left\{ -2 + \frac{(-2)(-2-1)}{2!} \cdot \frac{\delta x}{x} + \dots \right\} \right]$$

$$\frac{dy}{dx} = 2x - \frac{2}{x^3}$$

$$\boxed{\frac{d}{dx} \left(x^2 + \frac{1}{x^2} \right) = 2x - \frac{2}{x^3}} \quad \text{Ans.}$$

(ix) $(x+4)^{1/3}$ (Guj. Board 2003)

Let $y = (x+4)^{1/3}$

$$y + \delta y = (x + \delta x + 4)^{1/3}$$

$$\delta y = (x + 4 + \delta x)^{1/3} - y$$

$$\delta y = \left[(x+4) \left(1 + \frac{\delta x}{x+4} \right) \right]^{1/3} - (x+4)^{1/3} \quad \because y = (x+4)^{1/3}$$

$$\delta y = (x+4)^{1/3} \left(1 + \frac{\delta x}{x+4} \right)^{1/3} - (x+4)^{1/3}$$

$$\delta y = (x+4)^{1/3} \left[\left(1 + \frac{\delta x}{x+4} \right)^{1/3} - 1 \right]$$

$$\delta y = (x+4)^{1/3} \left[1 + \frac{1}{3} \left(\frac{\delta x}{x+4} \right) + \frac{1/3 (1/3 - 1)}{2!} \left(\frac{\delta x}{x+4} \right)^2 + \dots - 1 \right]$$



$$\delta y = (x+4)^{1/3} \cdot \frac{\delta x}{x+4} \left[\frac{1}{3} + \frac{1/3(1/3-1)}{2!} \cdot \frac{\delta x}{x+4} + \dots \right]$$

Dividing both sides by δx .

$$\frac{\delta y}{\delta x} = \frac{\delta x}{\delta x(x+4)} \cdot \frac{1}{x^{1-1/3}} \left[\frac{1}{3} + \frac{1/3(1/3-1)}{2!} \cdot \frac{\delta x}{x+4} + \dots \right]$$

$$\frac{\delta y}{\delta x} = \frac{1}{(x+4)^{2/3}} \left[\frac{1}{3} + \frac{1/3(1/3-1)}{2!} \cdot \frac{\delta x}{x+4} + \dots \right]$$

Taking limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{1}{(x+4)^{2/3}} \left[\frac{1}{3} + \frac{1/3(1/3-1)}{2!} \cdot \frac{\delta x}{x+4} + \dots \right]$$

$$\frac{dy}{dx} = \frac{1}{3(x+4)^{2/3}}$$

$$\boxed{\frac{d}{dx} [(x+4)^{1/3}] = \frac{1}{3(x+4)^{2/3}}}$$

Ans.

(x) $x^{3/2}$

Let

$$y = x^{3/2}$$

$$y + \delta y = (x + \delta x)^{3/2}$$

$$\delta y = (x + \delta x)^{3/2} - y$$

$$\delta y = \left[x \left(1 + \frac{\delta x}{x} \right) \right]^{3/2} - x^{3/2} \quad \because y = x^{3/2}$$

$$\delta y = x^{3/2} \left(1 + \frac{\delta x}{x} \right)^{3/2} - x^{3/2}$$

$$\delta y = x^{3/2} \left[\left(1 + \frac{\delta x}{x} \right)^{3/2} - 1 \right]$$

$$\delta y = x^{3/2} \left[1 + \frac{3}{2} \left(\frac{\delta x}{x} \right) + \frac{3/2 (3/2 - 1)}{2!} \cdot \left(\frac{\delta x}{x} \right)^2 + \dots - 1 \right]$$

$$\delta y = x^{3/2} \cdot \frac{\delta x}{x} \left[\frac{3}{2} + \frac{3/2 (3/2 - 1)}{2!} \cdot \frac{\delta x}{x} \right]$$

Dividing both sides by δx .

$$\frac{\delta y}{\delta x} = \frac{x^{\frac{3}{2}-1} \cdot \delta x}{\delta x} \left[\frac{3}{2} + \frac{3/2 (3/2 - 1)}{2!} \cdot \frac{\delta x}{x} + \dots \right]$$

$$\frac{\delta y}{\delta x} = x^{1/2} \left[\frac{3}{2} + \frac{3/2 (3/2 - 1)}{2!} \cdot \frac{\delta x}{x} + \dots \right]$$

Taking limit $\delta x \rightarrow 0$



$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} x^{1/2} \left[\frac{3}{2} + \frac{3/2 (3/2 - 1)}{2!} \cdot \frac{\delta x}{x} + \dots \right]$$

$$\frac{dy}{dx} = x^{1/2} \cdot \frac{3}{2}$$

$$\boxed{\frac{d}{dx} (x^{1/3}) = \frac{3}{2} \sqrt{x}} \quad \text{Ans.}$$

(xi) $x^{5/2}$

Let

$$y = x^{5/2}$$

$$y + \delta y = (x + \delta x)^{5/2}$$

$$\delta y = (x + \delta x)^{5/2} - y$$

$$\delta y = \left[x \left(1 + \frac{\delta x}{x} \right) \right]^{5/2} - x^{5/2} \quad \because y = x^{5/2}$$

$$\delta y = x^{5/2} \left(1 + \frac{\delta x}{x} \right)^{5/2} - x^{5/2}$$

$$\delta y = x^{5/2} \left[\left(1 + \frac{\delta x}{x} \right)^{5/2} - 1 \right]$$

$$\delta y = \left[1 + \frac{5}{2} \left(\frac{\delta x}{x} \right) + \frac{5/2 (5/2 - 1)}{2!} \cdot \left(\frac{\delta x}{x} \right)^2 + \dots - 1 \right]$$

$$\delta y = x^{5/2} \cdot \frac{\delta x}{x} \left[\frac{5}{2} + \frac{5/2 (5/2 - 1)}{2!} \cdot \frac{\delta x}{x} + \dots \right]$$

Dividing both sides by δx .

$$\frac{\delta y}{\delta x} = \frac{x^{\frac{5}{2}-1} \cdot \delta x}{\delta x} \left[\frac{5}{2} + \frac{5/2 (5/2 - 1)}{2!} \cdot \frac{\delta x}{x} + \dots \right]$$

$$\frac{\delta y}{\delta x} = x^{3/2} \left[\frac{5}{2} + \frac{5/2 (5/2 - 1)}{2!} \cdot \frac{\delta x}{x} + \dots \right]$$

Taking limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} x^{3/2} \left[\frac{5}{2} + \frac{5/2 (5/2 - 1)}{2!} \cdot \frac{\delta x}{x} + \dots \right]$$

$$\frac{dy}{dx} = \frac{5}{2} x^{3/2}$$

$$\boxed{\frac{d}{dx} (x^{5/2}) = \frac{5}{2} x^{3/2}} \quad \text{Ans.}$$

(xii) x^m 

Let $y = x^m$

$$y + \delta y = (x + \delta x)^m$$

$$\delta y = \left[x \left(1 + \frac{\delta x}{x} \right) \right]^m - y$$

$$\delta y = x^m \left(1 + \frac{\delta x}{x} \right)^m - x^m \quad \because y = x^m$$

$$\delta y = x^m \left[\left(1 + \frac{\delta x}{x} \right)^m - 1 \right]$$

$$\delta y = x^m \left[1 + m \left(\frac{\delta x}{x} \right) + \frac{m(m-1)}{2!} \cdot \left(\frac{\delta x}{x} \right)^2 + \dots - 1 \right]$$

$$\delta y = x^m \cdot \frac{\delta x}{x} \left[m + \frac{m(m-1)}{2!} \cdot \frac{\delta x}{x} + \dots \right]$$

Dividing both sides by δx .

$$\frac{\delta y}{\delta x} = \frac{x^{m-1} \delta x}{\delta x} \left[m + \frac{m(m-1)}{2!} \cdot \frac{\delta x}{x} + \dots \right]$$

$$\frac{\delta y}{\delta x} = x^{m-1} \left[m + \frac{m(m-1)}{2!} \cdot \frac{\delta x}{x} + \dots \right]$$

Taking limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} x^{m-1} \left[m + \frac{m(m-1)}{2!} \cdot \frac{\delta x}{x} + \dots \right]$$

$$\frac{dy}{dx} = mx^{m-1}$$

$\frac{d}{dx} (x^m) = mx^{m-1}$

Ans.

(xiii) $\frac{1}{x^m}, m \in \mathbb{N}$

Let $y = \frac{1}{x^m}$

$$y = x^{-m}$$

$$y + \delta y = (x + \delta x)^{-m}$$

$$\delta y = \left[x \left(1 + \frac{\delta x}{x} \right) \right]^{-m} - y$$

$$\delta y = x^{-m} \left(1 + \frac{\delta x}{x} \right)^{-m} - x^{-m} \quad \because y = x^{-m}$$



$$\begin{aligned}\delta y &= x^{-m} \left[\left(1 + \frac{\delta x}{x} \right)^{-m} - 1 \right] \\ \delta y &= x^{-m} \left[1 + (-m) \left(\frac{\delta x}{x} \right) + \frac{(-m)(-m-1)}{2!} \cdot \left(\frac{\delta x}{x} \right)^2 + \dots - 1 \right] \\ &= x^{-m} \cdot \frac{\delta x}{x} \left[-m + \frac{(-m)(-m-1)}{2!} \cdot \frac{\delta x}{x} + \dots \right]\end{aligned}$$

Dividing both sides by δx .

$$\begin{aligned}\frac{\delta y}{\delta x} &= \frac{x^{-m-1} \delta x}{\delta x} \left[-m + \frac{(-m)(-m-1)}{2!} \cdot \frac{\delta x}{x} + \dots \right] \\ \frac{\delta y}{\delta x} &= x^{-m-1} \left[-m + \frac{(-m)(-m-1)}{2!} \cdot \frac{\delta x}{x} + \dots \right]\end{aligned}$$

Taking limit $\delta x \rightarrow 0$

$$\begin{aligned}\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} &= \lim_{\delta x \rightarrow 0} x^{-m-1} \left[-m + \frac{(-m)(-m-1)}{2!} \cdot \frac{\delta x}{x} + \dots \right] \\ \frac{dy}{dx} &= -mx^{-m-1}\end{aligned}$$

$$\boxed{\frac{dy}{dx} \left(\frac{1}{x^m} \right) = -mx^{-m-1}}$$

Ans.

(xiv) x^{40}

Let

$$\begin{aligned}y &= x^{40} \\ y + \delta y &= (x + \delta x)^{40} \\ \delta y &= \left[x \left(1 + \frac{\delta x}{x} \right) \right]^{40} - y \\ \delta y &= x^{40} \left(1 + \frac{\delta x}{x} \right)^{40} - x^{40} \quad \because y = x^{40} \\ \delta y &= x^{40} \left[\left(1 + \frac{\delta x}{x} \right)^{40} - 1 \right] \\ \delta y &= x^{40} \left[1 + 40 \left(\frac{\delta x}{x} \right) + \frac{40(40-1)}{2!} \cdot \left(\frac{\delta x}{x} \right)^2 + \dots - 1 \right] \\ \delta y &= x^{40} \cdot \frac{\delta x}{x} \left[40 + \frac{40(40-1)}{2!} \cdot \frac{\delta x}{x} + \dots \right]\end{aligned}$$

Dividing both sides by δx .

$$\frac{\delta y}{\delta x} = \frac{x^{40-1} \delta x}{\delta x} \left[40 + \frac{40(40-1)}{2!} \cdot \frac{\delta x}{x} + \dots \right]$$



$$\frac{\delta y}{\delta x} = x^{39} \left[40 + \frac{40(40-1)}{2!} \cdot \frac{\delta x}{x} + \dots \right]$$

Taking limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} x^{39} \left[40 + \frac{40(40-1)}{2!} \cdot \frac{\delta x}{x} + \dots \right]$$

$$\frac{dy}{dx} = 40x^{39}$$

$$\boxed{\frac{dy}{dx} (x^{40}) = 40x^{39}}$$

Ans.

(xv) x^{-100}

Let

$$y = x^{-100}$$

$$y + \delta y = (x + \delta x)^{-100}$$

$$\delta y = \left[x \left(1 + \frac{\delta x}{x} \right) \right]^{-100} - y$$

$$\delta y = x^{-100} \left(1 + \frac{\delta x}{x} \right)^{-100} - x^{-100} \quad \because y = x^{-100}$$

$$\delta y = x^{-100} \left[\left(1 + \frac{\delta x}{x} \right)^{-100} - 1 \right]$$

$$\delta y = x^{-100} \left[1 + (-100) \left(\frac{\delta x}{x} \right) + \frac{(-100)(-100-1)}{2!} \cdot \left(\frac{\delta x}{x} \right)^2 + \dots - 1 \right]$$

$$\delta y = x^{-100} \cdot \frac{\delta x}{x} \left[-100 + \frac{(-100)(-100-1)}{2!} \cdot \frac{\delta x}{x} + \dots \right]$$

Dividing both sides by δx .

$$\frac{\delta y}{\delta x} = \frac{\delta x}{\delta x x^{1+100}} \left[-100 + \frac{(-100)(-100-1)}{2!} \cdot \frac{\delta x}{x} + \dots \right]$$

$$\frac{\delta y}{\delta x} = \frac{1}{x^{101}} \left[-100 + \frac{(-100)(-100-1)}{2!} \cdot \frac{\delta x}{x} + \dots \right]$$

Taking limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{1}{x^{101}} \left[-100 + \frac{(-100)(-100-1)}{2!} \cdot \frac{\delta x}{x} + \dots \right]$$

$$\frac{dy}{dx} = \frac{-100}{x^{101}}$$

$$\boxed{\frac{d}{dx} (x^{-100}) = \frac{-100}{x^{101}}}$$

Ans.



Q.2 Find $\frac{dy}{dx}$ from first principles if

(i) $\sqrt{x+2}$ (ii) $\frac{1}{\sqrt{x+a}}$ (L.B 2004, 2010)

Solution:

(i) $\sqrt{x+2}$

Let

$$y = \sqrt{x+2}$$

$$y = (x+2)^{1/2}$$

$$y + \delta y = (x + \delta x + 2)^{1/2}$$

$$\delta y = (x + 2 + \delta x)^{1/2} - y$$

$$\delta y = \left[(x+2) \left(1 + \frac{\delta x}{x+2} \right) \right]^{1/2} - (x+2)^{1/2} \quad \because y = \sqrt{x+2}$$

$$\delta y = (x+2)^{1/2} \left(1 + \frac{\delta x}{x+2} \right)^{1/2} - (x+2)^{1/2}$$

$$\delta y = (x+2)^{1/2} \left[\left(1 + \frac{\delta x}{x+2} \right)^{1/2} - 1 \right]$$

$$\delta y = (x+2)^{1/2} \left[1 + \frac{1}{2} \left(\frac{\delta x}{x+2} \right) + \frac{1/2(1/2-1)}{2!} \cdot \left(\frac{\delta x}{x+2} \right)^2 + \dots - 1 \right]$$

$$\delta y = (x+2)^{1/2} \cdot \frac{\delta x}{x+2} \left[\frac{1}{2} + \frac{1/2(1/2-1)}{2!} \cdot \frac{\delta x}{x+2} + \dots \right]$$

Dividing both sides by δx .

$$\frac{\delta y}{\delta x} = \frac{\delta x}{\delta x (x+2)^{1/2}} \left[\frac{1}{2} + \frac{1/2(1/2-1)}{2!} \cdot \frac{\delta x}{x+2} + \dots \right]$$

$$\frac{\delta y}{\delta x} = \frac{1}{(x+2)^{1/2}} \left[\frac{1}{2} + \frac{1/2(1/2-1)}{2!} \cdot \frac{\delta x}{x+2} + \dots \right]$$

Taking limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{1}{\sqrt{x+2}} \left[\frac{1}{2} + \frac{1/2(1/2-1)}{2!} \cdot \frac{\delta x}{x+2} + \dots \right]$$

$$\frac{\delta y}{\delta x} = \frac{1}{2\sqrt{x+2}}$$

$$\boxed{\frac{d}{dx}(\sqrt{x+2}) = \frac{1}{2\sqrt{x+2}}}$$

Ans.



(ii) $\frac{1}{\sqrt{x+a}}$

Let $y = \frac{1}{\sqrt{x+a}}$

$$y = (x+a)^{-1/2}$$

$$y + \delta y = (x + \delta x + a)^{-1/2}$$

$$\delta y = (x + a + \delta x)^{-1/2} - y$$

$$\delta y = \left[(x+a) \left(1 + \frac{\delta x}{x+a} \right) \right]^{-1/2} - (x+a)^{-1/2} \quad \because y = (x+a)^{-1/2}$$

$$\delta y = (x+a)^{-1/2} \left(1 + \frac{\delta x}{x+a} \right)^{-1/2} - (x+a)^{-1/2}$$

$$\delta y = (x+a)^{-1/2} \left[\left(1 + \frac{\delta x}{x+a} \right)^{-1/2} - 1 \right]$$

$$\delta y = (x+a)^{-1/2} \left[1 + \left(\frac{1}{2} \right) \left(\frac{\delta x}{x+a} \right) + \frac{(-1/2)(-1/2-1)}{2!} \cdot \left(\frac{\delta x}{x+a} \right)^2 + \dots - 1 \right]$$

$$\delta y = (x+a)^{-1/2} \cdot \frac{\delta x}{x+a} \left[\frac{-1}{2} + \frac{(-1/2)(-1/2-1)}{2!} \cdot \frac{\delta x}{x+a} + \dots \right]$$

Dividing both sides by δx .

$$\frac{\delta y}{\delta x} = \frac{\delta x}{\delta x (x+a)^{1+1/2}} \left[\frac{-1}{2} + \frac{-1/2(-1/2-1)}{2!} \cdot \frac{\delta x}{x+a} + \dots \right]$$

$$\frac{\delta y}{\delta x} = \frac{1}{(x+a)^{3/2}} \left[\frac{-1}{2} + \frac{-1/2(-1/2-1)}{2!} \cdot \frac{\delta x}{x+a} + \dots \right]$$

Taking limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{1}{(x+a)^{3/2}} \left[\frac{-1}{2} + \frac{-1/2(-1/2-1)}{2!} \cdot \frac{\delta x}{x+a} + \dots \right]$$

$$\frac{dy}{dx} = \frac{-1}{2(x+a)^{3/2}}$$

$$\boxed{\frac{d}{dx} \left(\frac{1}{\sqrt{x+a}} \right) = \frac{-1}{2(x+a)^{3/2}}}$$

Ans.



EXERCISE 2.2

Q. Find from first principles, the derivatives of the following expressions w.r.t. their respective independent variables.

(i) $(ax + b)^3$ (L.B 2004)

(ii) $(2x + 3)^5$

(iii) $(3t + 2)^{-2}$

(iv) $(ax + b)^{-5}$

(v) $\frac{1}{(az - b)^7}$ (L.B 2010)

Solution:

(i) $(ax + b)^3$

Let

$$y = (ax + b)^3$$

$$y + \delta y = [a(x + \delta x) + b]^3$$

$$\delta y = (ax + a\delta x + b)^3 - y$$

$$\delta y = (ax + b + a\delta x)^3 - (ax + b)^3 \quad \because y = (ax + b)^3$$

$$\delta y = \left[(ax + b) \left(1 + \frac{a\delta x}{ax + b} \right) \right]^3 - (ax + b)^3$$

$$\delta y = (ax + b)^3 \left(1 + \frac{a\delta x}{ax + b} \right)^3 - (ax + b)^3$$

$$\delta y = (ax + b)^3 \left[\left(1 + \frac{a\delta x}{ax + b} \right)^3 - 1 \right]$$

$$\delta y = (ax + b)^3 \left[1 + 3 \left(\frac{a\delta x}{ax + b} \right) + \frac{3(3-1)}{2!} \cdot \left(\frac{a\delta x}{ax + b} \right)^2 + \dots - 1 \right]$$

$$\delta y = (ax + b)^3 \cdot \frac{a\delta x}{ax + b} \left[3 + \frac{3(3-1)}{2!} \cdot \frac{a\delta x}{ax + b} + \dots \right]$$

Dividing both sides by δx .

$$\frac{\delta y}{\delta x} = \frac{(ax + b)^{3-1} \cdot a\delta x}{\delta x} \left[3 + \frac{3(3-1)}{2!} \cdot \frac{a\delta x}{ax + b} + \dots \right]$$

$$\frac{\delta y}{\delta x} = a(ax + b)^2 \left[3 + \frac{3(3-1)}{2!} \cdot \frac{a\delta x}{ax + b} + \dots \right]$$

Taking limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} a(ax + b)^2 \left[3 + \frac{3(3-1)}{2!} \cdot \frac{a\delta x}{ax + b} + \dots \right]$$

$$\frac{dy}{dx} = 3a(ax + b)^2$$



$$\boxed{\frac{d}{dx} (ax + b)^3 = 3a(ax + b)^2} \quad \text{Ans.}$$

(ii) $(2x + 3)^5$

Let $y = (2x + 3)^5$

$$y + \delta y = [2(x + \delta x) + 3]^5$$

$$\delta y = (2x + 2\delta x + 3)^5 - y$$

$$\delta y = (2x + 3 + 2\delta x)^5 - (2x + 3)^5 \quad \because y = (2x + 3)^5$$

$$\delta y = \left[(2x + 3) \left(1 + \frac{2\delta x}{2x + 3} \right) \right]^5 - (2x + 3)^5$$

$$\delta y = (2x + 3)^5 \left(1 + \frac{2\delta x}{2x + 3} \right)^5 - (2x + 3)^5$$

$$\delta y = (2x + 3)^5 \left[\left(1 + \frac{2\delta x}{2x + 3} \right)^5 - 1 \right]$$

$$\delta y = (2x + 3)^5 \left[1 + 5 \left(\frac{2\delta x}{2x + 3} \right) + \frac{5(5-1)}{2!} \cdot \left(\frac{2\delta x}{2x + 3} \right)^2 + \dots - 1 \right]$$

$$\delta y = (2x + 3)^5 \frac{2\delta x}{2x + 3} \left[5 + \frac{5(5-1)}{2!} \cdot \frac{2\delta x}{2x + 3} + \dots \right]$$

Dividing both sides by δx .

$$\frac{\delta y}{\delta x} = \frac{2\delta x (2x + 3)^{5-1}}{\delta x} \left[5 + \frac{5(5-1)}{2!} \cdot \frac{2\delta x}{2x + 3} + \dots \right]$$

$$\frac{\delta y}{\delta x} = 2(2x + 3)^4 \left[5 + \frac{5(5-1)}{2!} \cdot \frac{2\delta x}{2x + 3} + \dots \right]$$

Taking limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} 2(2x + 3)^4 \left[5 + \frac{5(5-1)}{2!} \cdot \frac{2\delta x}{2x + 3} + \dots \right]$$

$$\frac{dy}{dx} = 2(2x + 3)^4 \cdot 5$$

$$\boxed{\frac{d}{dx} (2x + 3)^5 = 10(2x + 3)^4} \quad \text{Ans.}$$

(iii) $(3t + 2)^{-2}$

Let $y = (3t + 2)^{-2}$

$$y + \delta y = [3(t + \delta t) + 2]^{-2}$$

$$\delta y = (3t + 3\delta t + 2)^{-2} - y$$

$$\delta y = (3t + 2 + 3\delta t)^{-2} - (3t + 2)^{-2} \quad \because y = (3t + 2)^{-2}$$



$$\delta y = \left[(3t+2) \left(1 + \frac{3\delta t}{3t+2} \right) \right]^{-2} - (3t+2)^{-2}$$

$$\delta y = (3t+2)^{-2} \left(1 + \frac{3\delta t}{3t+2} \right)^{-2} - (3t+2)^{-2}$$

$$\delta y = (3t+2)^{-2} \left[\left(1 + \frac{3\delta t}{3t+2} \right)^{-2} - 1 \right]$$

$$\delta y = (3t+2)^{-2} \left[1 + (-2) \left(\frac{3\delta t}{3t+2} \right) + \frac{(-2)(-2-1)}{2!} \cdot \left(\frac{3\delta t}{3t+2} \right)^2 + \dots - 1 \right]$$

$$\delta y = (3t+2)^{-2} \frac{3\delta t}{3t+2} \left[-2 + \frac{(-2)(-2-1)}{2!} \cdot \frac{3\delta t}{3t+2} + \dots \right]$$

Dividing both sides by δt .

$$\frac{\delta y}{\delta t} = \frac{3\delta t}{\delta t(3t+2)^{1+2}} \left[-2 + \frac{(-2)(-2-1)}{2!} \cdot \frac{3\delta t}{3t+2} + \dots \right]$$

$$\frac{\delta y}{\delta t} = \frac{3}{(3t+2)^3} \left[-2 + \frac{(-2)(-2-1)}{2!} \cdot \frac{3\delta t}{3t+2} + \dots \right]$$

Taking limit $\delta t \rightarrow 0$

$$\lim_{\delta t \rightarrow 0} \frac{\delta y}{\delta t} = \lim_{\delta t \rightarrow 0} \frac{3}{(3t+2)^3} \left[-2 + \frac{(-2)(-2-1)}{2!} \cdot \frac{3\delta t}{3t+2} + \dots \right]$$

$$\frac{dy}{dt} = \frac{3}{(3t+2)^3} (-2)$$

$$\boxed{\frac{d}{dt} (3t+2)^{-2} = \frac{-6}{(3t+2)^3}} \quad \text{Ans.}$$

(iv) $(ax+b)^{-5}$

Let $y = (ax+b)^{-5}$

$$y + \delta y = [a(x + \delta x) + b]^{-5}$$

$$\delta y = (ax + a\delta x + b)^{-5} - y$$

$$\delta y = (ax + b + a\delta x)^{-5} - (ax + b)^{-5} \quad \because y = (ax + b)^{-5}$$

$$\delta y = \left[(ax + b) \left(1 + \frac{a\delta x}{ax + b} \right) \right]^{-5} - (ax + b)^{-5}$$

$$\delta y = (ax + b)^{-5} \left(1 + \frac{a\delta x}{ax + b} \right)^{-5} - (ax + b)^{-5}$$

$$\delta y = (ax + b)^{-5} \left[\left(1 + \frac{a\delta x}{ax + b} \right)^{-5} - 1 \right]$$



$$\delta y = (ax+b)^{-5} \left[1 + (-5) \left(\frac{a\delta x}{ax+b} \right) + \frac{(-5)(-5-1)}{2!} \cdot \left(\frac{a\delta x}{ax+b} \right)^2 + \dots -1 \right]$$

$$\delta y = (ax+b)^{-5} \frac{a\delta x}{ax+b} \left[-5 + \frac{(-5)(-5-1)}{2!} \cdot \frac{a\delta x}{ax+b} + \dots \right]$$

Dividing both sides by δx .

$$\frac{\delta y}{\delta x} = \frac{a\delta x}{\delta x(ax+b)^{1+5}} \left[-5 + \frac{(-5)(-5-1)}{2!} \cdot \frac{a\delta x}{ax+b} + \dots \right]$$

$$\frac{\delta y}{\delta x} = \frac{a}{(ax+b)^6} \left[-5 + \frac{(-5)(-5-1)}{2!} \cdot \frac{a\delta x}{ax+b} + \dots \right]$$

Taking limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{a}{(ax+b)^6} \left[-5 + \frac{(-5)(-5-1)}{2!} \cdot \frac{a\delta x}{ax+b} + \dots \right]$$

$$\frac{dy}{dx} = \frac{a}{(ax+b)^6} (-5)$$

$$\boxed{\frac{d}{dt}(ax+b)^{-5} = \frac{-5a}{(ax+b)^6}} \quad \text{Ans.}$$

(v) $\frac{1}{(az-b)^7}$

Let $y = \frac{1}{(az-b)^7}$

$$y = (az-b)^{-7}$$

$$y + \delta y = [a(z + \delta z) - b]^{-7}$$

$$\delta y = (az + a\delta z - b)^{-7} - y$$

$$\delta y = (az - b + a\delta z)^{-7} - (az - b)^{-7} \quad \because y = (az - b)^{-7}$$

$$\delta y = \left[(az - b) \left(1 + \frac{a\delta z}{az - b} \right) \right]^{-7} - (az - b)^{-7}$$

$$\delta y = (az - b)^{-7} \left(1 + \frac{a\delta z}{az - b} \right)^{-7} - (az - b)^{-7}$$

$$\delta y = (az - b)^{-7} \left[\left(1 + \frac{a\delta z}{az - b} \right)^{-7} - 1 \right]$$

$$\delta y = (az - b)^{-7} \left[1 + (-7) \left(\frac{a\delta z}{az - b} \right) + \frac{(-7)(-7-1)}{2!} \cdot \left(\frac{a\delta z}{az - b} \right)^2 + \dots -1 \right]$$

$$\delta y = (az - b)^{-7} \frac{a\delta z}{az - b} \left[-7 + \frac{(-7)(-7-1)}{2!} \cdot \frac{a\delta z}{az - b} + \dots \right]$$

Dividing both sides by δz .



$$\frac{\delta y}{\delta z} = \frac{a\delta z}{\delta z(az - b)^{1+7}} \left[-7 + \frac{(-7)(-7-1)}{2!} \cdot \frac{a\delta z}{az - b} + \dots \right]$$

$$\frac{\delta y}{\delta z} = \frac{a}{(az - b)^8} \left[-7 + \frac{(-7)(-7-1)}{2!} \cdot \frac{a\delta z}{az - b} + \dots \right]$$

Taking limit $\delta z \rightarrow 0$

$$\lim_{\delta z \rightarrow 0} \frac{\delta y}{\delta z} = \lim_{\delta z \rightarrow 0} \frac{a}{(az - b)^8} \left[-7 + \frac{(-7)(-7-1)}{2!} \cdot \frac{a\delta z}{az - b} + \dots \right]$$

$$\frac{dy}{dz} = \frac{a}{(az - b)^8} (-7)$$

$$\boxed{\frac{d}{dz} \left[\frac{1}{(az - b)^7} \right] = \frac{-7a}{(az - b)^8}}$$

Ans.



EXERCISE 2.3

Q.1 Differentiate w.r.t. 'x'

$$x^4 + 2x^3 + x^2$$

Solution:

Let $y = x^4 + 2x^3 + x^2$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) + 2 \frac{d}{dx}(x^3) + \frac{d}{dx}(x^2)$$

$$\frac{dy}{dx} = 4x^{4-1} \cdot \frac{d}{dx}(x) + 2 \cdot 3x^{3-1} \cdot \frac{d}{dx}(x) + 2x^{2-1} \cdot \frac{d}{dx}(x)$$

$$\frac{dy}{dx} = 4x^3 \cdot 1 + 6x^2 \cdot 1 + 2x \cdot 1$$

$$\boxed{\frac{dy}{dx} = 4x^3 + 6x^2 + 2x} \quad \text{Ans.}$$

Q.2 $x^{-3} + 2x^{-3/2} + 2$

Solution:

Let $y = x^{-3} + 2x^{-3/2} + 2$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(x^{-3}) + 2 \frac{d}{dx}(x^{-3/2}) + \frac{d}{dx}(2)$$



$$\frac{dy}{dx} = -3x^{-3-1} \cdot \frac{d}{dx}(x) + 2 \cdot \frac{-3}{2} x^{\frac{3}{2}-1} \cdot \frac{d}{dx}(x) + 0$$

$$\frac{dy}{dx} = -3x^{-4} \cdot 1 - 3x^{-5/2} \cdot 1$$

$$\boxed{\frac{dy}{dx} = -3 \left(\frac{1}{x^4} + \frac{1}{x^{5/2}} \right)} \quad \text{Ans.}$$

Q.3 $\frac{a+x}{a-x}$ (Lhr. Board 2008, 2010, 2011)

Solution:

Let $y = \frac{a+x}{a-x}$

Diff. w.r.t. 'x'

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left(\frac{a+x}{a-x} \right) \\ &= \frac{(a-x) \frac{d}{dx}(a+x) - (a+x) \frac{d}{dx}(a-x)}{(a-x)^2} \\ &= \frac{(a-x)(0+1) - (a+x)(0-1)}{(a-x)^2} \\ &= \frac{a-x+a+x}{(a-x)^2} \end{aligned}$$

$$\boxed{\frac{dy}{dx} = \frac{2a}{(a-x)^2}} \quad \text{Ans.}$$

Q.4 $\frac{2x-3}{2x+1}$ (L.B 2008)

Solution:

Let $y = \frac{2x-3}{2x+1}$

Diff. w.r.t. 'x'

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left(\frac{2x-3}{2x+1} \right) \\ \frac{dy}{dx} &= \frac{(2x+1) \frac{d}{dx}(2x-3) - (2x-3) \frac{d}{dx}(2x+1)}{(2x+1)^2} \end{aligned}$$



$$\frac{dy}{dx} = \frac{(2x+1) \cdot 2 - (2x-3) \cdot 2}{(2x+1)^2}$$

$$\frac{dy}{dx} = \frac{2[2x+1-2x+3]}{(2x+1)^2}$$

$$\frac{dy}{dx} = \frac{2(4)}{(2x+1)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{8}{(2x+1)^2}} \quad \text{Ans.}$$

Q.5 $(x-5)(3-x)$

Solution:

Let $y = (x-5)(3-x)$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx} [(x-5)(3-x)]$$

$$\frac{dy}{dx} = (x-5) \frac{d}{dx} (3-x) + (3-x) \frac{d}{dx} (x-5)$$

$$\frac{dy}{dx} = (x-5)(-1) + (3-x)(1)$$

$$\frac{dy}{dx} = -x + 5 + 3 - x$$

$$\boxed{\frac{dy}{dx} = 8 - 2x} \quad \text{Ans.}$$

Q.6 $\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2$

Solution:

Let $y = \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2$

$$y = (\sqrt{x})^2 + \left(\frac{1}{\sqrt{x}}\right)^2 - 2(\sqrt{x})\left(\frac{1}{\sqrt{x}}\right)$$

$$y = x + \frac{1}{x} - 2$$

$$y = x + x^{-1} - 2$$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx} (x) + \frac{d}{dx} (x^{-1}) - \frac{d}{dx} (2)$$



$$\frac{dy}{dx} = 1 + (-1) x^{-1-1} \cdot \frac{d}{dx}(x) - 0$$

$$\frac{dy}{dx} = 1 - x^{-2} \cdot 1$$

$$\frac{dy}{dx} = 1 - \frac{1}{x^2}$$

$$\boxed{\frac{dy}{dx} = \frac{x^2 - 1}{x^2}} \quad \text{Ans.}$$

Q.7 $\frac{(1 + \sqrt{x})(x - x^{3/2})}{\sqrt{x}}$

Solution:

$$y = \frac{(1 + \sqrt{x})(x - x^{3/2})}{\sqrt{x}}$$

$$y = \frac{(1 + \sqrt{x})(x - x\sqrt{x})}{\sqrt{x}}$$

$$y = \frac{x(1 + \sqrt{x})(1 - \sqrt{x})}{\sqrt{x}}$$

$$y = \frac{\sqrt{x} \times \sqrt{x} [(1)^2 - (\sqrt{x})^2]}{\sqrt{x}}$$

$$y = \sqrt{x} (1 - x)$$

$$y = \sqrt{x} - x\sqrt{x}$$

$$y = x^{1/2} - x^{3/2}$$

Diff. w.r.t 'x'

$$\frac{dy}{dx} = \frac{dy}{dx}(x^{1/2}) - \frac{d}{dx}(x^{3/2})$$

$$\frac{dy}{dx} = \frac{1}{2} x^{1/2-1} \cdot \frac{d}{dx}(x) - \frac{3}{2} x^{3/2-1} \cdot \frac{d}{dx}(x)$$

$$\frac{dy}{dx} = \frac{1}{2} x^{1/2-1} \cdot 1 - \frac{3}{2} x^{3/2-1} \cdot 1$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} - \frac{3\sqrt{x}}{2}$$

$$\boxed{\frac{dy}{dx} = \frac{1 - 3x}{2\sqrt{x}}} \quad \text{Ans.}$$



Q.8 $\frac{(x^2 + 1)^2}{x^2 - 1}$ (L.B 2007) (G.B 2007)

Solution:

$$y = \frac{(x^2 + 1)^2}{x^2 - 1}$$

Diff. w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx} \left[\frac{(x^2 + 1)^2}{x^2 - 1} \right]$$

$$\frac{dy}{dx} = \frac{(x^2 - 1) \frac{d}{dx} (x^2 + 1)^2 - (x^2 + 1)^2 \cdot \frac{d}{dx} (x^2 - 1)}{(x^2 - 1)^2}$$

$$\frac{dy}{dx} = \frac{(x^2 - 1) \cdot 2(x^2 + 1) \cdot \frac{d}{dx} (x^2 + 1) - (x^2 + 1)^2 \cdot 2x}{(x^2 - 1)^2}$$

$$\frac{dy}{dx} = \frac{2(x^2 - 1)(x^2 + 1) \cdot 2x - 2x(x^2 + 1)^2}{(x^2 - 1)^2}$$

$$\frac{dy}{dx} = \frac{2x(x^2 + 1) [2(x^2 - 1) - (x^2 + 1)]}{(x^2 - 1)^2}$$

$$\frac{dy}{dx} = \frac{2x(x^2 + 1)(2x^2 - 2 - x^2 - 1)}{(x^2 - 1)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{2x(x^2 + 1)(x^2 - 3)}{(x^2 - 1)^2}} \quad \text{Ans.}$$

Q.9 $\frac{x^2 + 1}{x^2 - 3}$ (L.B 2009)

Solution:

Let $y = \frac{x^2 + 1}{x^2 - 3}$

Diff. w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{x^2 + 1}{x^2 - 3} \right)$$

$$\frac{dy}{dx} = \frac{(x^2 - 3) \frac{d}{dx} (x^2 + 1) - (x^2 + 1) \frac{d}{dx} (x^2 - 3)}{(x^2 - 3)^2}$$

$$\frac{dy}{dx} = \frac{(x^2 - 3) \cdot 2x - (x^2 + 1) \cdot 2x}{(x^2 - 3)^2}$$



$$\frac{dy}{dx} = \frac{2x(x^2 - 3 - x^2 - 1)}{(x^2 - 3)^2}$$

$$\frac{dy}{dx} = \frac{2x(-4)}{(x^2 - 3)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{-8x}{(x^2 - 3)^2}} \quad \text{Ans.}$$

Q.10 $\frac{\sqrt{1+x}}{\sqrt{1-x}}$

Solution:

Let $y = \frac{\sqrt{1+x}}{\sqrt{1-x}}$

$$y = \left(\frac{1+x}{1-x}\right)^{1/2}$$

Diff. w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{1+x}{1-x}\right)^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}-1} \cdot \frac{d}{dx} \left(\frac{1+x}{1-x}\right)$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{1+x}{1-x}\right)^{-1/2} \left[\frac{(1-x) \frac{d}{dx} (1+x) - (1+x) \frac{d}{dx} (1-x)}{(1-x)^2} \right]$$

$$\frac{dy}{dx} = \frac{1}{2} \frac{(1+x)^{-1/2}}{(1-x)^{-1/2}} \left[\frac{(1-x)(0+1) - (1+x)(0-1)}{(1-x)^2} \right]$$

$$\frac{dy}{dx} = \frac{1-x+1+x}{2(1+x)^{1/2} (1-x)^{-1/2+2}}$$

$$\frac{dy}{dx} = \frac{2}{2\sqrt{1+x} (1-x)^{3/2}}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{\sqrt{1+x} (1-x)^{3/2}}} \quad \text{Ans.}$$

Q.11 $\frac{2x-1}{\sqrt{x^2+1}}$

Solution:

Let $y = \frac{2x-1}{\sqrt{x^2+1}}$



Diff. w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{2x-1}{\sqrt{x^2+1}} \right)$$

$$\frac{dy}{dx} = \frac{(\sqrt{x^2+1}) \frac{d}{dx} (2x-1) - (2x-1) \frac{d}{dx} (\sqrt{x^2+1})}{(\sqrt{x^2+1})^2}$$

$$\frac{dy}{dx} = \frac{(\sqrt{x^2+1}) 2 - (2x-1) \cdot \frac{1}{2} (x^2+1)^{-1/2} \cdot 2x}{x^2+1}$$

$$\frac{dy}{dx} = \frac{2\sqrt{x^2+1} - \frac{x(2x-1)}{\sqrt{x^2+1}}}{x^2+1}$$

$$\frac{dy}{dx} = \frac{\frac{2(x^2+1) - 2x^2 + x}{\sqrt{x^2+1}}}{x^2+1}$$

$$\frac{dy}{dx} = \frac{2x^2 + 2 - 2x^2 + x}{(x^2+1)\sqrt{x^2+1}}$$

$$\boxed{\frac{dy}{dx} = \frac{x+2}{(x^2+1)^{3/2}}} \text{ Ans.}$$

Q.12 $\sqrt{\frac{a-x}{a+x}}$ (L.B 2004)

Solution:

Let $y = \sqrt{\frac{a-x}{a+x}}$

$$y = \left(\frac{a-x}{a+x} \right)^{1/2}$$

Diff. w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{a-x}{a+x} \right)^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{a-x}{a+x} \right)^{-1/2} \cdot \frac{d}{dx} \left(\frac{a-x}{a+x} \right)$$

$$\frac{dy}{dx} = \frac{1}{2} \frac{(a-x)^{-1/2}}{(a+x)^{-1/2}} \left[\frac{(a+x) \frac{d}{dx} (a-x) - (a-x) \frac{d}{dx} (a+x)}{(a+x)^2} \right]$$



$$\frac{dy}{dx} = \frac{(a+x)(0-1) - (a-x)(0+1)}{2(a-x)^{1/2} (a+x)^{-\frac{1}{2}+2}}$$

$$\frac{dy}{dx} = \frac{-a-x-a+x}{2\sqrt{a-x} (a+x)^{3/2}}$$

$$\frac{dy}{dx} = \frac{-2a}{2\sqrt{a-x} (a+x)^{3/2}}$$

$$\boxed{\frac{dy}{dx} = \frac{-a}{\sqrt{a-x} (a+x)^{3/2}}} \quad \text{Ans.}$$

Q.13 $\frac{\sqrt{x^2+1}}{\sqrt{x^2-1}}$

Solution:

Let $y = \frac{\sqrt{x^2+1}}{\sqrt{x^2-1}} \quad y = \left(\frac{x^2+1}{x^2-1}\right)^{1/2}$

Diff. w.r.t 'x'.

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{x^2+1}{x^2-1}\right)^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{x^2+1}{x^2-1}\right)^{1/2} \cdot \frac{d}{dx} \left(\frac{x^2+1}{x^2-1}\right)^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{x^2+1}{x^2-1}\right)^{-1/2} \left[\frac{(x^2-1) \frac{d}{dx} (x^2+1) - (x^2+1) \frac{d}{dx} (x^2-1)}{(x^2-1)^2} \right]$$

$$\frac{dy}{dx} = \frac{(x^2-1) \cdot 2x - (x^2+1) \cdot 2x}{2(x^2+1)^{1/2} (x^2-1)^{-1/2+2}}$$

$$\frac{dy}{dx} = \frac{2x(x^2-1-x^2-1)}{2\sqrt{x^2+1} (x^2-1)^{3/2}}$$

$$\boxed{\frac{dy}{dx} = \frac{-2x}{\sqrt{x^2+1} (x^2-1)^{3/2}}} \quad \text{Ans.}$$

Q.14 $\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$

Solution:

Let $y = \frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$



$$\begin{aligned}
 y &= \frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \times \frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} - \sqrt{1-x}} \\
 y &= \frac{(\sqrt{1+x} - \sqrt{1-x})^2}{(\sqrt{1+x})^2 - (\sqrt{1-x})^2} \\
 y &= \frac{(\sqrt{1+x})^2 + (\sqrt{1-x})^2 - 2(\sqrt{1+x})(\sqrt{1-x})}{1+x - (1-x)} \\
 y &= \frac{1+x + 1-x - 2\sqrt{(1+x)(1-x)}}{1+x - 1+x} \\
 y &= \frac{2 - 2\sqrt{1-x^2}}{2x} \\
 y &= \frac{2(1 - \sqrt{1-x^2})}{2x} \\
 y &= \frac{1 - \sqrt{1-x^2}}{x}
 \end{aligned}$$

Diff. w.r.t. 'x'

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx} \left(\frac{1 - \sqrt{1-x^2}}{x} \right) \\
 \frac{dy}{dx} &= \frac{x \frac{d}{dx} (1 - \sqrt{1-x^2}) - (1 - \sqrt{1-x^2}) \frac{d}{dx} (x)}{x^2} \\
 \frac{dy}{dx} &= \frac{x \cdot \frac{-1}{2} (1-x^2)^{-1/2} \cdot -2x - (1 - \sqrt{1-x^2}) \cdot 1}{x^2} \\
 \frac{dy}{dx} &= \frac{\frac{x^2}{\sqrt{1-x^2}} - 1 + \sqrt{1-x^2}}{x^2} \\
 \frac{dy}{dx} &= \frac{x^2 - \sqrt{1-x^2} + 1 - x^2}{x^2 \sqrt{1-x^2}}
 \end{aligned}$$

$$\boxed{\frac{dy}{dx} = \frac{1 - \sqrt{1-x^2}}{x^2 \sqrt{1-x^2}}} \quad \text{Ans.}$$

Q.15 $\frac{x\sqrt{a+x}}{\sqrt{a-x}}$



Solution:

Let $y = \frac{x\sqrt{a+x}}{\sqrt{a-x}} \quad y = x \left(\frac{a+x}{a-x} \right)^{1/2}$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx} \left[x \left(\frac{a+x}{a-x} \right)^{1/2} \right]$$

$$= x \frac{d}{dx} \left(\frac{a+x}{a-x} \right)^{1/2} + \left(\frac{a+x}{a-x} \right)^{1/2} \cdot \frac{d}{dx} (x)$$

$$\frac{dy}{dx} = x \cdot \frac{1}{2} \left(\frac{a+x}{a-x} \right)^{-1/2} \cdot \frac{d}{dx} \left(\frac{a+x}{a-x} \right) + \frac{\sqrt{a+x}}{\sqrt{a-x}}$$

$$\frac{dy}{dx} = \frac{x}{2} \frac{(a+x)^{-1/2}}{(a-x)^{-1/2}} \left[\frac{(a-x) \frac{d}{dx} (a+x) - (a+x) \frac{d}{dx} (a-x)}{(a-x)^2} \right] + \frac{\sqrt{a+x}}{\sqrt{a-x}}$$

$$\frac{dy}{dx} = \frac{x [(a-x)(0+1) - (a+x)(0-1)]}{2(a+x)^{1/2} (a-x)^{\frac{-1}{2}+2}} + \frac{\sqrt{a+x}}{\sqrt{a-x}}$$

$$\frac{dy}{dx} = \frac{x(a-x+a+x)}{2\sqrt{a+x} (a-x)^{3/2}} + \frac{\sqrt{a+x}}{\sqrt{a-x}}$$

$$\frac{dy}{dx} = \frac{2ax}{2\sqrt{a+x} (a-x)^{3/2}} + \frac{\sqrt{a+x}}{\sqrt{a-x}}$$

$$\frac{dy}{dx} = \frac{ax}{\sqrt{a+x} (a-x)^{3/2}} + \frac{\sqrt{a+x}}{\sqrt{a-x}}$$

$$\frac{dy}{dx} = \frac{ax + (a+x)(a-x)}{\sqrt{a+x} (a-x)^{3/2}}$$

$$\frac{dy}{dx} = \frac{ax + a^2 - x^2}{\sqrt{a+x} (a-x)^{3/2}}$$

$$\boxed{\frac{dy}{dx} = \frac{a^2 + ax - x^2}{\sqrt{a+x} (a-x)^{3/2}}} \quad \text{Ans.}$$

Q.16 If $y = \sqrt{x} - \frac{1}{\sqrt{x}}$, show that $2x \frac{dy}{dx} + y = 2\sqrt{x}$. (L.B 2004)

Solution:

Let $y = \sqrt{x} - \frac{1}{\sqrt{x}}$

$$y = x^{1/2} - x^{-1/2}$$



Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx} (x^{1/2}) - \frac{d}{dx} (x^{-1/2})$$

$$\frac{dy}{dx} = \frac{1}{2} x^{-1/2} - \left(-\frac{1}{2}\right) x^{-3/2}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} + \frac{1}{2x^{3/2}}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} + \frac{1}{2x\sqrt{x}}$$

Taking $2x \frac{dy}{dx} + y = 2x \left(\frac{1}{2\sqrt{x}} + \frac{1}{2x\sqrt{x}} \right) + \sqrt{x} - \frac{1}{\sqrt{x}}$

$$= \frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}} + \sqrt{x} - \frac{1}{\sqrt{x}}$$

$$= \sqrt{x} + \sqrt{x}$$

$$\boxed{2x \frac{dy}{dx} + y = 2\sqrt{x}} \quad \text{Hence proved.}$$

Q.17 If $y = x^4 + 2x^2 + 2$, prove that $\frac{dy}{dx} = 4x\sqrt{y-1}$. (G.B 2006)(L.B 2009(S) (L.B 2007)

Solution:

Let $y = x^4 + 2x^2 + 2$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx} (x^4) + 2 \frac{d}{dx} (x^2) + \frac{d}{dx} (2)$$

$$= 4x^3 + 2(2x) + 0$$

$$\frac{dy}{dx} = 4x^3 + 4x$$

Taking,

$$4x\sqrt{y-1} = 4x\sqrt{x^4 + 2x^2 + 2 - 1}$$

$$= 4x\sqrt{x^4 + 2x^2 + 1}$$

$$= 4x\sqrt{(x^2 + 1)^2}$$

$$= 4x(x^2 + 1)$$

$$= 4x^3 + 4x$$

$$\frac{dy}{dx} = 4x\sqrt{y-1} \quad \text{Hence proved.}$$



EXERCISE 2.4

Q.1 Find $\frac{dy}{dx}$ by making suitable substitutions in the following functions defined as:

(i) $y = \sqrt{\frac{1-x}{1+x}}$

(ii) $y = \sqrt{x + \sqrt{x}}$

(iii) $y = \sqrt{\frac{a+x}{a-x}}$

(iv) $y = (3x^2 - 2x + 7)^6$

(v) $y = \sqrt{\frac{a^2 + x^2}{a^2 - x^2}}$

Solution:

(i) $y = \sqrt{\frac{1-x}{1+x}}$

Let $u = \frac{1-x}{1+x}$ So $y = \sqrt{u} = u^{1/2}$

Diff. w.r.t. 'x'

$$\frac{du}{dx} = \frac{d}{dx} \left(\frac{1-x}{1+x} \right)$$

$$\frac{du}{dx} = \frac{(1+x) \frac{d}{dx} (1-x) - (1-x) \frac{d}{dx} (1+x)}{(1+x)^2}$$

$$\frac{du}{dx} = \frac{(1+x)(0-1) - (1-x)(0+1)}{(1+x)^2}$$

$$\frac{du}{dx} = \frac{-1-x-1+x}{(1+x)^2}$$

$$\frac{du}{dx} = \frac{-2}{(1+x)^2}$$

$$y = u^{1/2}$$

Diff. w.r.t. 'u'.

$$\frac{dy}{du} = \frac{d}{du} (u^{1/2})$$

$$\frac{dy}{du} = \frac{1}{2} u^{-1/2}$$

$$\frac{dy}{du} = \frac{1}{2} \left(\frac{1-x}{1+x} \right)^{-1/2}$$



$$\frac{dy}{du} = \frac{1}{2} \left(\frac{1+x}{1-x} \right)^{1/2}$$

$$\frac{dy}{du} = \frac{(1+x)^{1/2}}{2(1-x)^{1/2}}$$

By using chain rule.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{(1+x)^{1/2}}{2(1-x)^{1/2}} \cdot \frac{-2}{(1+x)^2}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x} (1+x)^{2-\frac{1}{2}}}$$

$$\boxed{\frac{dy}{dx} = \frac{-1}{\sqrt{1-x} (1+x)^{3/2}}}$$

Ans.

(ii) $y = \sqrt{x + \sqrt{x}}$ (G.B 2007)

Let

$$u = x + \sqrt{x} \quad \text{So } y = \sqrt{u} = u^{1/2}$$

Diff. w.r.t. 'x'

$$\frac{du}{dx} = \frac{d}{dx}(x) + \frac{d}{dx}(x^{1/2})$$

$$\frac{du}{dx} = 1 + \frac{1}{2} x^{-1/2}$$

$$\frac{du}{dx} = 1 + \frac{1}{2\sqrt{x}}$$

$$\frac{du}{dx} = \frac{2\sqrt{x} + 1}{2\sqrt{x}}$$

$$y = u^{1/2}$$

Diff. w.r.t. 'u'

$$\frac{dy}{du} = \frac{d}{du}(u^{1/2})$$

$$\frac{dy}{du} = \frac{1}{2} u^{-1/2} \cdot 1$$

$$\frac{dy}{du} = \frac{1}{2} (x + \sqrt{x})^{-1/2}$$

$$\frac{dy}{du} = \frac{1}{2\sqrt{x + \sqrt{x}}}$$



By using chain rule.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ \frac{dy}{dx} &= \frac{1}{2\sqrt{x+\sqrt{x}}} \cdot \frac{2\sqrt{x}+1}{2\sqrt{x}} \\ \frac{dy}{dx} &= \frac{2\sqrt{x}+1}{4\sqrt{x}\sqrt{x+\sqrt{x}}} \quad \text{Ans.}\end{aligned}$$

(iii) $y = x \sqrt{\frac{a+x}{a-x}}$

Put,

$$u = \frac{a+x}{a-x} \quad \text{So} \quad y = x\sqrt{u} = xu^{1/2}$$

Now,

$$u = xu^{1/2}$$

Diff. w.r.t. 'x'.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(xu^{1/2}) \\ \frac{dy}{dx} &= x \frac{d}{dx}(u^{1/2}) + u^{1/2} \cdot \frac{d}{dx}(x) \\ \frac{dy}{dx} &= \frac{x}{2} u^{-1/2} \frac{du}{dx} + u^{1/2} \quad \dots\dots\dots (1)\end{aligned}$$

$$u = \frac{a+x}{a-x}$$

Diff. w.r.t. 'x'.

$$\begin{aligned}\frac{du}{dx} &= \frac{d}{dx}\left(\frac{a+x}{a-x}\right) \\ \frac{du}{dx} &= \frac{(a-x) \frac{d}{dx}(a+x) - (a+x) \frac{d}{dx}(a-x)}{(a-x)^2} \\ \frac{du}{dx} &= \frac{(a-x)(0+1) - (a+x)(0-1)}{(a-x)^2} \\ \frac{du}{dx} &= \frac{a-x+a+x}{(a-x)^2} \\ \frac{du}{dx} &= \frac{2a}{(a-x)^2}\end{aligned}$$



∴ From equation (1).

$$\frac{dy}{dx} = \frac{x}{2} \left(\frac{a+x}{a-x} \right)^{-1/2} \cdot \frac{2a}{(a-x)^2} + \left(\frac{a+x}{a-x} \right)^{1/2}$$

$$\frac{dy}{dx} = ax \frac{(a+x)^{-1/2}}{(a-x)^{-1/2}} \cdot \frac{1}{(a-x)^2} + \frac{\sqrt{a+x}}{\sqrt{a-x}}$$

$$\frac{dy}{dx} = \frac{ax}{\sqrt{a+x} (a-x)^{-1/2+2}} + \frac{\sqrt{a+x}}{\sqrt{a-x}}$$

$$\frac{dy}{dx} = \frac{ax}{\sqrt{a+x} (a-x)^{3/2}} + \frac{\sqrt{a+x}}{\sqrt{a-x}}$$

$$\frac{dy}{dx} = \frac{ax + (a+x)(a-x)}{\sqrt{a+x} (a-x)^{3/2}}$$

$$\frac{dy}{dx} = \frac{ax + a^2 - x^2}{\sqrt{a+x} (a-x)^{3/2}}$$

$$\boxed{\frac{dy}{dx} = \frac{a^2 - x^2 + ax}{\sqrt{a+x} (a-x)^{3/2}}} \quad \text{Ans.}$$

(iv) $y = (3x^2 - 2x + 7)^6$ (L.B 2009 (s))

Let $u = 3x^2 - 2x + 7$ So $y = u^6$

Diff. w.r.t. 'x'.

$$\frac{du}{dx} = 3 \frac{d}{dx} (x^2) - 2 \frac{d}{dx} (x) + \frac{d}{dx} (7)$$

$$= 3(2x) - 2(1) + 0$$

$$\frac{du}{dx} = 6x - 2$$

$$y = u^6$$

Diff. w.r.t. 'u'.

$$\frac{dy}{du} = \frac{d}{du} (u^6)$$

$$\frac{dy}{du} = 6u^5$$

$$\frac{dy}{du} = 6(3x^2 - 2x + 7)^5$$

By using chain rule.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$



$$\frac{dy}{dx} = 6(3x^2 - 2x + 7)^5 \cdot (6x - 2)$$

$$\frac{dy}{dx} = 12(3x^2 - 2x + 7)^5 (3x - 1)$$

Ans.

(v) $y = \sqrt{\frac{a^2 + x^2}{a^2 - x^2}}$ (G.B 2004)

Let $u = \frac{a^2 + x^2}{a^2 - x^2}$ So $y = \sqrt{u} = u^{1/2}$

Diff. w.r.t. 'x'.

$$\frac{du}{dx} = \frac{d}{dx} \left(\frac{a^2 + x^2}{a^2 - x^2} \right)$$

$$\frac{du}{dx} = \frac{(a^2 - x^2) \frac{d}{dx} (a^2 + x^2) - (a^2 + x^2) \frac{d}{dx} (a^2 - x^2)}{(a^2 - x^2)^2}$$

$$\frac{du}{dx} = \frac{(a^2 - x^2)2x - (a^2 + x^2) \cdot (-2x)}{(a^2 - x^2)^2}$$

$$\frac{du}{dx} = \frac{2x(a^2 - x^2 + a^2 + x^2)}{(a^2 - x^2)^2}$$

$$\frac{du}{dx} = \frac{2x(2a^2)}{(a^2 - x^2)^2} = \frac{4a^2x}{(a^2 - x^2)^2}$$

$$y = u^{1/2}$$

Diff. w.r.t. 'u'.

$$\frac{dy}{du} = \frac{d}{du} (u^{-1/2})$$

$$\frac{dy}{du} = \frac{1}{2} u^{-1/2}$$

$$\frac{dy}{du} = \frac{1}{2} \left(\frac{a^2 + x^2}{a^2 - x^2} \right)^{-1/2}$$

By using chain rule.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{a^2 + x^2}{a^2 - x^2} \right)^{-1/2} \cdot \frac{4a^2x}{(a^2 - x^2)^2}$$

$$\frac{dy}{dx} = \frac{(a^2 + x^2)^{-1/2}}{(a^2 - x^2)^{-1/2}} \cdot \frac{2a^2x}{(a^2 - x^2)^2}$$



$$\frac{dy}{dx} = \frac{2a^2x}{(a^2 + x^2)^{1/2} (a^2 - x^2)^{-1/2+2}}$$

$$\boxed{\frac{dy}{dx} = \frac{2a^2x}{\sqrt{a^2 + x^2} (a^2 - x^2)^{3/2}}} \quad \text{Ans.}$$

Q.2 Find $\frac{dy}{dx}$ if:

(i) $3x + 4y + 7 = 0$

(ii) $xy + y^2 = 2$

(iii) $x^2 - 4xy - 5y = 0$

(iv) $4x^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

(v) $x\sqrt{1+y} + y\sqrt{1+x} = 0$

(vi) $y(x^2 - 1) = x\sqrt{x^2 + 4}$

Solution:

(i) $3x + 4y + 7 = 0$

Diff. w.r.t 'x'.

$$3 \frac{d}{dx}(x) + 4 \frac{dy}{dx} + \frac{d}{dx}(7) = 0$$

$$3.1 + 4 \frac{dy}{dx} = 0$$

$$4 \frac{dy}{dx} = -3$$

$$\boxed{\frac{dy}{dx} = \frac{-3}{4}} \quad \text{Ans.}$$

(ii) $xy + y^2 = 2$

Diff. w.r.t 'x'.

$$\frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(2)$$

$$x \frac{dy}{dx} + y \frac{d}{dx}(x) + 2y \frac{dy}{dx} = 0$$

$$(x + 2y) \frac{dy}{dx} = -y$$

$$\boxed{\frac{dy}{dx} = \frac{-y}{x + 2y}} \quad \text{Ans.}$$

(iii) $x^2 - 4xy - 5y = 0$

Diff. w.r.t 'x'.

$$\frac{d}{dx}(x^2) - 4 \frac{d}{dx}(xy) - 5 \frac{dy}{dx} = 0$$



$$2x - 4 \left[x \frac{dy}{dx} + y \cdot 1 \right] - 5 \frac{dy}{dx} = 0$$

$$2x - 4x \frac{dy}{dx} - 4y - 5 \frac{dy}{dx} = 0$$

$$-(4x + 5) \frac{dy}{dx} = -2x + 4y$$

$$\frac{dy}{dx} = \frac{-2(x - 2y)}{-(4x + 5)}$$

$$\boxed{\frac{dy}{dx} = \frac{2(x - 2y)}{4x + 5}}$$

Ans.

(iv) $4x^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Diff. w.r.t 'x'

$$4 \frac{d}{dx} (x^2) + 2h \frac{d}{dx} (xy) + b \frac{d}{dx} (y^2) + 2g \frac{d}{dx} (x) + 2f \frac{dy}{dx} + \frac{d}{dx} (c) = 0$$

$$4(2x) + 2h \left[x \frac{dy}{dx} + y \cdot 1 \right] + b \cdot 2y \frac{dy}{dx} + 2g \cdot 1 + 2f \frac{dy}{dx} + 0 = 0$$

$$8x + 2hx \frac{dy}{dx} + 2hy + 2by \frac{dy}{dx} + 2g + 2f \frac{dy}{dx} = 0$$

$$2(hx + by + f) \frac{dy}{dx} = -8x - 2hy - 2g$$

$$\frac{dy}{dx} = \frac{-2(4x + hy + g)}{2(hx + by + f)}$$

$$\boxed{\frac{dy}{dx} = -\frac{4x + hy + g}{hx + by + f}}$$

Ans.

(v) $x\sqrt{1+y} + y\sqrt{1+x} = 0$ (L.B 2007) (G.B 2007)

Diff. w.r.t 'x'.

$$\frac{d}{dx} (x\sqrt{1+y}) + \frac{d}{dx} (y\sqrt{1+x}) = 0$$

$$x \frac{d}{dx} (\sqrt{1+y}) + \sqrt{1+y} \frac{d}{dx} (x) + y \frac{d}{dx} (\sqrt{1+x}) + \sqrt{1+x} \frac{dy}{dx} = 0$$

$$x \cdot \frac{1}{2} (1+y)^{-1/2} \frac{dy}{dx} + \sqrt{1+y} \cdot 1 + y \cdot \frac{1}{2} (1+x)^{-1/2} \cdot 1 + \sqrt{1+x} \frac{dy}{dx} = 0$$

$$\left(\frac{x}{2\sqrt{1+y}} + \sqrt{1+x} \right) \frac{dy}{dx} + \sqrt{1+y} + \frac{y}{2\sqrt{1+x}} = 0$$

$$\left(\frac{x + 2\sqrt{1+x}\sqrt{1+y}}{2\sqrt{1+y}} \right) \frac{dy}{dx} = -\sqrt{1+y} - \frac{y}{2\sqrt{1+x}}$$



$$\frac{dy}{dx} = \frac{2\sqrt{1+y}}{x+2\sqrt{1+x}\sqrt{1+y}} \left[\frac{-2\sqrt{1+x}\sqrt{1+y}-y}{2\sqrt{1+x}} \right]$$

$$\frac{dy}{dx} = -\frac{\sqrt{1+y}(y+2\sqrt{1+x}\sqrt{1+y})}{\sqrt{1+x}(x+2\sqrt{1+x}\sqrt{1+y})} \text{ Ans.}$$

(vi) $y(x^2 - 1) = x\sqrt{x^2 + 4}$

Diff. w.r.t 'x'.

$$\frac{d}{dx} [y(x^2 - 1)] = \frac{d}{dx} (x\sqrt{x^2 + 4})$$

$$y \frac{d}{dx} (x^2 - 1) + (x^2 - 1) \frac{dy}{dx} = x \frac{d}{dx} (\sqrt{x^2 + 4}) + \sqrt{x^2 + 4} \frac{d}{dx} (x)$$

$$y \cdot 2x + (x^2 - 1) \frac{dy}{dx} = x \frac{1}{2} (x^2 + 4)^{-1/2} \cdot 2x + \sqrt{x^2 + 4}$$

$$(x^2 - 1) \frac{dy}{dx} = \frac{x^2}{\sqrt{x^2 + 4}} + \sqrt{x^2 + 4} - 2xy$$

$$\frac{dy}{dx} = \frac{1}{x^2 - 1} \left[\frac{x^2 + x^2 + 4 - 2xy\sqrt{x^2 + 4}}{\sqrt{x^2 + 4}} \right]$$

$$\frac{dy}{dx} = \frac{2x^2 + 4 - 2xy\sqrt{x^2 + 4}}{(x^2 - 1)\sqrt{x^2 + 4}}$$

$$\frac{dy}{dx} = \frac{2x^2 + 4 - 2x\sqrt{x^2 + 4} \cdot \frac{x\sqrt{x^2 + 4}}{x^2 - 1}}{(x^2 - 1)\sqrt{x^2 + 4}} \quad \because y = \frac{x\sqrt{x^2 + 4}}{x^2 - 1}$$

$$\frac{dy}{dx} = \frac{2x^2 + 4 - \frac{2x^2(x^2 + 4)}{x^2 - 1}}{(x^2 - 1)\sqrt{x^2 + 4}}$$

$$\frac{dy}{dx} = \frac{\frac{2x^2(x^2 - 1) + 4(x^2 - 1) - 2x^2(x^2 + 4)}{x^2 - 1}}{(x^2 - 1)\sqrt{x^2 + 4}}$$

$$\frac{dy}{dx} = \frac{2x^4 - 2x^2 + 4x^2 - 4 - 2x^4 - 8x^2}{(x^2 - 1)^2 \sqrt{x^2 + 4}}$$

$$= \frac{-6x^2 - 4}{(x^2 - 1)^2 \sqrt{x^2 + 4}}$$

$$\boxed{\frac{dy}{dx} = \frac{-2(3x^2 + 2)}{(x^2 - 1)^2 \sqrt{x^2 + 4}}} \quad \text{Ans.}$$



Q.3 Find $\frac{dy}{dx}$ of the following parametric functions.

(i) $x = \theta + \frac{1}{\theta}$ and $y = \theta + 1$ (ii) $x = \frac{a(1-t^2)}{1+t^2}$ and $y = \frac{2bt}{1+t^2}$

Solution:

(i) $x = \theta + \frac{1}{\theta}$ and $y = \theta + 1$

$$\begin{aligned} x &= \theta + \theta^{-1} \\ \text{Diff. w.r.t. '}\theta\text{' } \\ \frac{dx}{d\theta} &= \frac{d}{d\theta}(\theta) + \frac{d}{d\theta}(\theta^{-1}) \\ \frac{dx}{d\theta} &= 1 + (-1)\theta^{-2} \\ \frac{dx}{d\theta} &= 1 - \frac{1}{\theta^2} \\ \frac{dx}{d\theta} &= \frac{\theta^2 - 1}{\theta^2} \end{aligned}$$

$$\begin{aligned} y &= \theta + 1 \\ \text{Diff. w.r.t. '}\theta\text{' } \\ \frac{dy}{d\theta} &= \frac{d}{d\theta}(\theta) + \frac{d}{d\theta}(1) \\ \frac{dy}{d\theta} &= 1 + 0 = 1 \end{aligned}$$

By using chain rule.

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} \Rightarrow \frac{dy}{dx} = 1 \cdot \frac{\theta^2}{\theta^2 - 1}$$

$$\boxed{\frac{dy}{dx} = \frac{\theta^2}{\theta^2 - 1}} \quad \text{Ans.}$$

(ii) $x = \frac{a(1-t^2)}{1+t^2}$ and $y = \frac{2bt}{1+t^2}$

$$x = \frac{a(1-t^2)}{1+t^2}$$

Diff. w.r.t. 't'

$$\begin{aligned} \frac{dx}{dt} &= a \frac{d}{dt} \left(\frac{1-t^2}{1+t^2} \right) \\ \frac{dx}{dt} &= a \left[\frac{(1+t^2) \frac{d}{dt}(1-t^2) - (1-t^2) \frac{d}{dt}(1+t^2)}{(1+t^2)^2} \right] \\ \frac{dx}{dt} &= a \left[\frac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2} \right] \\ \frac{dx}{dt} &= \frac{a \cdot 2t(-1-t^2-1+t^2)}{(1+t^2)^2} \\ \frac{dx}{dt} &= \frac{2at(-2)}{(1+t^2)^2} \end{aligned}$$

$$y = \frac{2bt}{1+t^2}$$

Diff. w.r.t. 't'

$$\begin{aligned} \frac{dy}{dt} &= 2b \frac{d}{dt} \left[\frac{t}{1+t^2} \right] \\ \frac{dy}{dt} &= 2b \left[\frac{(1+t^2) \frac{d}{dt}(t) - t \frac{d}{dt}(1+t^2)}{(1+t^2)^2} \right] \\ \frac{dy}{dt} &= 2b \left[\frac{(1+t^2) - t \cdot 2t}{(1+t^2)^2} \right] \\ \frac{dy}{dt} &= \frac{2b(1+t^2-2t^2)}{(1+t^2)^2} \\ \frac{dy}{dt} &= \frac{2b(1-t^2)}{(1+t^2)^2} \end{aligned}$$



$$\frac{dx}{dt} = \frac{-4at}{(1+t^2)^2}$$

By using chain rule.

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{2b(1-t^2)}{(1+t^2)^2} \times \frac{(1+t^2)^2}{-4at}$$

$$\frac{dy}{dx} = -\frac{b(1-t^2)}{2at} \quad \text{Ans.}$$

Q.4: Prove that $y \frac{dy}{dx} + x = 0$ if $x = \frac{1-t^2}{1+t^2}$, $y = \frac{2t}{1+t^2}$ (Guj. Board 2005, 2008)

Solution:

$$x = \frac{1-t^2}{1+t^2}, \quad y = \frac{2t}{1+t^2}$$

$$x = \frac{1-t^2}{1+t^2}$$

Diff. w.r.t. 't'

$$\frac{dx}{dt} = \frac{d}{dt} \left(\frac{1-t^2}{1+t^2} \right)$$

$$\frac{dx}{dt} = \frac{(1+t^2) \frac{d}{dt} (1-t^2) - (1-t^2) \frac{d}{dt} (1+t^2)}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{2t(-1-t^2-1+t^2)}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{2t(-2)}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{-4t}{(1+t^2)^2}$$

$$y = \frac{2t}{1+t^2}$$

Diff. w.r.t 't'

$$\frac{dy}{dt} = 2 \frac{d}{dt} \left(\frac{t}{1+t^2} \right)$$



$$\begin{aligned}\frac{dy}{dt} &= 2 \left[\frac{(1+t^2) \frac{d}{dt}(t) - t \frac{d}{dt}(1+t^2)}{(1+t^2)^2} \right] \\ \frac{dy}{dt} &= 2 \left[\frac{(1+t^2) \cdot 1 - t \cdot 2t}{(1+t^2)^2} \right] \\ \frac{dy}{dt} &= \frac{2(1+t^2-2t^2)}{(1+t^2)^2} \\ \frac{dy}{dt} &= \frac{2(1-t^2)}{(1+t^2)^2}\end{aligned}$$

By using chain rule

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{dt}{dx} \\ \frac{dy}{dx} &= \frac{2(1-t^2)}{(1+t^2)^2} \times \frac{(1+t^2)}{-4t} \\ \frac{dy}{dx} &= \frac{-(1-t^2)}{2t} = \frac{t^2-1}{2t}\end{aligned}$$

Taking

$$\begin{aligned}y \frac{dy}{dx} + x &= \frac{2t}{1+t^2} \left(\frac{t^2-1}{2t} \right) + \frac{1-t^2}{1+t^2} \\ &= \frac{t^2-1}{1+t^2} + \frac{1-t^2}{1+t^2} \\ &= \frac{t^2-1+1-t^2}{1+t^2} \\ &= \frac{0}{1+t^2} = 0 \quad \text{Hence proved.}\end{aligned}$$

Q.5: Differentiate

(i) $x^2 - \frac{1}{x^2}$ w.r.t. x^4

(ii) $(1+x^2)^n$ w.r.t. x^2

(iii) $\frac{x^2+1}{x^2-1}$ w.r.t. $\frac{x-1}{x+1}$

(iv) $\frac{ax+b}{cx+d}$ w.r.t. $\frac{ax^2+b}{ax^2+d}$

(v) $\frac{x^2+1}{x^2-1}$ w.r.t. x^3

Solution:

(i) $x^2 - \frac{1}{x^2}$ w.r.t. x^4 (L.B 2006)

Let $y = x^2 - \frac{1}{x^2}$, $u = x^4$

$y = x^2 - x^{-2}$

Diff. w.r.t. 'x'

$u = x^4$

Diff. w.r.t. 'x'



$$\begin{aligned}
 \frac{dy}{dx} &= 2x - (-2)x^{-3} \\
 &= 2x + \frac{2}{x^3} \\
 &= \frac{2x^4 + 2}{x^3} \\
 &= \frac{2(x^4 + 1)}{x^3}
 \end{aligned}
 \quad \left| \quad
 \begin{aligned}
 \frac{du}{dx} &= 4x^3
 \end{aligned}$$

By using chain rule

$$\begin{aligned}
 \frac{dy}{du} &= \frac{dy}{dx} \cdot \frac{dx}{du} \\
 \frac{dy}{du} &= \frac{2(x^4 + 1)}{x^3} \times \frac{1}{4x^3} \\
 \frac{dy}{du} &= \frac{x^4 + 1}{2x^6} \quad \text{Ans}
 \end{aligned}$$

(ii) $(1 + x^2)^n$ w.r.t. x^2

Let

$$\begin{aligned}
 y &= (1 + x^2)^n, & u &= x^2 \\
 y &= (1 + x^2)^n & u &= x^2 \\
 \text{Diff. w.r.t. 'x'} & & \text{Diff. w.r.t. 'x'} & \\
 \frac{dy}{dx} &= \frac{d}{dx} (1 + x^2)^n & \frac{du}{dx} &= \frac{d}{dx} x^2 \\
 &= n(1 + x^2)^{n-1} \cdot 2x & \frac{du}{dx} &= 2x \\
 &= 2nx(1 + x^2)^{n-1}
 \end{aligned}$$

By using chain rule

$$\begin{aligned}
 \frac{dy}{du} &= \frac{dy}{dx} \cdot \frac{dx}{du} \\
 &= 2nx(1 + x^2)^{n-1} \cdot \frac{1}{2x}
 \end{aligned}$$

$$\boxed{\frac{dy}{du} = n(1 + x^2)^{n-1}} \quad \text{Ans.}$$

(iii) $\frac{x^2 + 1}{x^2 - 1}$ w.r.t. $\frac{x-1}{x+1}$

Let $y = \frac{x^2 + 1}{x^2 - 1}, \quad u = \frac{x-1}{x+1}$



$$y = \frac{x^2 + 1}{x^2 - 1}$$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{x^2 + 1}{x^2 - 1} \right)$$

$$= \frac{(x^2 - 1) \frac{d}{dx}(x^2 + 1) - (x^2 + 1) \frac{d}{dx}(x^2 - 1)}{(x^2 - 1)^2}$$

$$= \frac{(x^2 - 1)(2x) - (x^2 + 1)(2x)}{(x^2 - 1)^2}$$

$$= \frac{2x(x^2 - 1 - x^2 - 1)}{(x^2 - 1)^2}$$

$$= \frac{2x(-2)}{(x^2 - 1)^2}$$

$$\frac{dy}{dx} = \frac{-4x}{(x^2 - 1)^2}$$

By using chain rule

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du}$$

$$= \frac{-4x}{(x^2 - 1)^2} \times \frac{(x + 1)^2}{2}$$

$$= \frac{-2x(x + 1)^2}{[(x + 1)(x - 1)]^2}$$

$$= \frac{-2x(x + 1)^2}{(x + 1)^2(x - 1)^2}$$

$$\boxed{\frac{dy}{du} = \frac{-2x}{(x - 1)^2}}$$

Ans.

(iv) $\frac{ax + b}{cx + d}$ w.r.t. $\frac{ax^2 + b}{ax^2 + d}$

Let $y = \frac{ax + b}{cx + d}$, $u = \frac{ax^2 + b}{ax^2 + d}$

$$y = \frac{ax + b}{cx + d}$$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{ax + b}{cx + d} \right)$$

$$u = \frac{x - 1}{x + 1}$$

Diff. w.r.t. 'x'

$$\frac{du}{dx} = \frac{d}{dx} \left(\frac{x - 1}{x + 1} \right)$$

$$= \frac{(x + 1) \frac{d}{dx}(x - 1) - (x - 1) \frac{d}{dx}(x + 1)}{(x + 1)^2}$$

$$= \frac{x + 1 - x + 1}{(x + 1)^2}$$

$$\frac{du}{dx} = \frac{2x}{(x + 1)^2}$$



$$\begin{aligned}
 &= \frac{(cx+d)\frac{d}{dx}(ax+b) - (ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2} \\
 &= \frac{(cx+d)a - (ax+b).c}{(cx+d)^2} \\
 &= \frac{acx + ad - acx - bc}{(cx+d)^2} \\
 \frac{dy}{dx} &= \frac{ad - bc}{(cx+d)^2}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{(ax^2+d)\frac{d}{dx}(ax^2+b) - (ax^2+b)\frac{d}{dx}(ax^2+d)}{(ax^2+d)^2} \\
 &= \frac{(ax^2+d)(2ax) - (ax^2+b)(2ax)}{(ax^2+d)^2} \\
 \frac{du}{dx} &= \frac{2ax(ax^2+d - ax^2 - b)}{(ax^2+d)^2} \\
 \frac{du}{dx} &= \frac{2ax(d-b)}{(ax^2+d)^2}
 \end{aligned}$$

By using chain rule

$$\begin{aligned}
 \frac{dy}{du} &= \frac{dy}{dx} \cdot \frac{dx}{du} \\
 &= \frac{ad - bc}{(cx+d)^2} \cdot \frac{(ax^2+d)^2}{2ax(d-b)}
 \end{aligned}$$

$$\boxed{\frac{dy}{du} = \frac{(ad - bc)(ax^2+d)^2}{2ax(cx+d)^2(d-b)}}$$

Ans.

(v) $\frac{x^2+1}{x^2-1}$ w.r.t. x^3 (G.B 2003)

Let

$$y = \frac{x^2+1}{x^2-1}, \quad u = x^3$$

$$y = \frac{x^2+1}{x^2-1}$$

Diff. w.r.t. 'x'

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx} \left(\frac{x^2+1}{x^2-1} \right) \\
 &= \frac{(x^2-1)\frac{d}{dx}(x^2+1) - (x^2+1)\frac{d}{dx}(x^2-1)}{(x^2-1)^2} \\
 &= \frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2} \\
 &= \frac{2x(x^2-1 - x^2-1)}{(x^2-1)^2} \\
 \frac{dy}{dx} &= \frac{2x(-2)}{(x^2-1)^2} \\
 &= \frac{-4x}{(x^2-1)^2}
 \end{aligned}$$

$$\begin{aligned}
 u &= x^3 \\
 \text{Diff. w.r.t. 'x'} \\
 \frac{du}{dx} &= \frac{d}{dx} (x^3) \\
 &= 3x^2
 \end{aligned}$$



By using chain rule

$$\frac{dx}{du} = \frac{dy}{dx} \cdot \frac{dx}{du}$$

$$\frac{dx}{du} = \frac{-4x}{(x^2 - 1)^2} \times \frac{1}{3x^2}$$

$$\frac{dx}{du} = \frac{-4x}{3x(x^2 - 1)^2} \quad \text{Ans.}$$

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EXERCISE 2.5

Q.1: Differentiate the following trigonometric functions from the first principles.

(i) $\sin 2x$

(ii) $\tan 3x$

(iii) $\sin 2x + \cos 2x$

(iv) $\cos x^2$

(v) $\tan^2 x$

(vi) $\sqrt{\tan x}$

(vii) $\cos \sqrt{x}$

Solution:

(i) $\sin 2x$ (L.B 2003)

Let $y = \sin 2x$

$$y + \delta y = \sin 2(x + \delta x)$$

$$\delta y = \sin (2x + 2\delta x) - y$$

$$\delta y = \sin (2x + 2\delta x) - \sin 2x$$

$$\delta y = 2 \cos \left(\frac{2x + 2\delta x + 2x}{2} \right) \cdot \sin \left(\frac{2x + 2\delta x - 2x}{2} \right) \quad \because y = \sin 2x$$

$$[\because \sin p - \sin q = 2 \cos \left(\frac{p+q}{2} \right) \sin \left(\frac{p-q}{2} \right)]$$

$$\delta y = 2 \cos \left(\frac{4x + 2\delta x}{2} \right) \cdot \sin \left(\frac{2\delta x}{2} \right)$$

$$\delta y = 2 \cos \left(\frac{4x + 2\delta x}{2} \right) \cdot \sin (\delta x)$$

Dividing both sides by δx

$$\frac{\delta y}{\delta x} = 2 \cos \left(\frac{4x + 2\delta x}{2} \right) \cdot \frac{\sin \delta x}{\delta x}$$

Taking limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} 2 \cos \left(\frac{4x + 2\delta x}{2} \right) \cdot \frac{\sin \delta x}{\delta x}$$



$$\frac{dy}{dx} = 2 \cos\left(\frac{4x}{2}\right) \cdot 1$$

$$\boxed{\frac{dy}{dx}(\sin 2x) = 2 \cos 2x}$$

Ans.

(ii) tan 3x

Let

$$y = \tan 3x$$

$$y + \delta y = \tan 3(x + \delta x)$$

$$\delta y = \tan(3x + 3\delta x) - y$$

$$\delta y = \tan(3x + 3\delta x) - \tan 3x \quad \because y = \tan 3x$$

$$\delta y = \frac{\sin(3x + 3\delta x)}{\cos(3x + 3\delta x)} - \frac{\sin 3x}{\cos 3x}$$

$$\delta y = \frac{\sin(3x + 3\delta x) \cos 3x - \cos(3x + 3\delta x) \sin 3x}{\cos(3x + 3\delta x) \cdot \cos 3x}$$

$$\delta y = \frac{\sin(3x + 3\delta x - 3x)}{\cos(3x + 3\delta x) \cos 3x} \quad [\because \sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta]$$

$$\delta y = \frac{\sin 3\delta x}{\cos(3x + 3\delta x) \cos 3x}$$

Dividing both sides by δx

$$\frac{\delta y}{\delta x} = \frac{3}{\cos(3x + 3\delta x) \cos 3x} \cdot \frac{\sin 3\delta x}{3\delta x}$$

Taking limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{3}{\cos(3x + 3\delta x) \cos 3x} \cdot \frac{\sin 3\delta x}{3\delta x}$$

$$\frac{dy}{dx} = \frac{3}{\cos 3x \cdot \cos 3x} \cdot 1$$

$$\frac{dy}{dx} = \frac{3}{\cos^2 3x}$$

$$\boxed{\frac{d}{dx}(\tan 3x) = 3 \sec^2 3x} \quad \text{Ans.}$$

(iii) sin 2x + cos 2x

Let

$$y = \sin 2x + \cos 2x$$

$$y + \delta y = \sin 2(x + \delta x) + \cos 2(x + \delta x)$$

$$\delta y = \sin(2x + 2\delta x) + \cos(2x + 2\delta x) - y$$

$$\delta y = \sin(2x + 2\delta x) + \cos(2x + 2\delta x) - (\sin 2x + \cos 2x)$$



$$\therefore y = \sin 2x + \cos 2x$$

$$\delta y = \sin(2x + 2\delta x) - \sin 2x + \cos(2x + 2\delta x) - \cos 2x$$

$$\delta y = 2 \cos\left(\frac{2x + 2\delta x + 2x}{2}\right) \cdot \sin\left(\frac{2x + 2\delta x - 2x}{2}\right) - 2 \sin\left(\frac{2x + 2\delta x + 2x}{2}\right) \sin\left(\frac{2x + 2\delta x - 2x}{2}\right)$$

$$\delta y = 2 \cos\left(\frac{4x + 2\delta x}{2}\right) \cdot \sin\left(\frac{2\delta x}{2}\right) - 2 \sin\left(\frac{4x + 2\delta x}{2}\right) \sin\left(\frac{2\delta x}{2}\right)$$

$$\delta y = 2 \sin \delta x \left[\cos\left(\frac{4x + 2\delta x}{2}\right) - \sin\left(\frac{4x + 2\delta x}{2}\right) \right]$$

Dividing both sides by δx

$$\frac{\delta y}{\delta x} = \frac{2 \sin \delta x}{\delta x} \left[\cos\left(\frac{4x + 2\delta x}{2}\right) - \sin\left(\frac{4x + 2\delta x}{2}\right) \right]$$

Taking limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} 2 \frac{\sin \delta x}{\delta x} \left[\cos\left(\frac{4x + 2\delta x}{2}\right) - \sin\left(\frac{4x + 2\delta x}{2}\right) \right]$$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \left[\cos\left(\frac{4x}{2}\right) - \sin\left(\frac{4x}{2}\right) \right]$$

$$\boxed{\frac{d}{dx}(\sin 2x + \cos 2x) = 2(\cos 2x - \sin 2x)} \quad \text{Ans.}$$

(iv) $\cos x^2$ (L.B 2003)

Let $y = \cos x^2$

$$y + \delta y = \cos(x + \delta x)^2$$

$$\delta y = \cos(x + \delta x)^2 - y$$

$$\delta y = \cos(x + \delta x)^2 - \cos x^2 \quad \therefore (y = \cos x^2)$$

$$\delta y = -2 \sin\left[\frac{(x + \delta x)^2 + x^2}{2}\right] \cdot \sin\left[\frac{(x + \delta x)^2 - x^2}{2}\right]$$

$$\delta y = -2 \sin\left[\frac{(x + \delta x)^2 + x^2}{2}\right] \cdot \sin\left(\frac{x^2 + \delta x^2 + 2x\delta x - x^2}{2}\right)$$

$$\delta y = -2 \sin\left[\frac{(x + \delta x)^2 + x^2}{2}\right] \cdot \sin\left[\frac{\delta x(\delta x + 2x)}{2}\right]$$

Dividing both sides by δx



$$\frac{\delta y}{\delta x} = -2 \sin \left[\frac{(x + \delta x)^2 + x^2}{2} \right] \frac{\sin \left[\frac{\delta x (\delta x + 2x)}{2} \right]}{\left(\frac{2}{\delta x + 2x} \right) \cdot \delta x \left(\frac{\delta x + 2x}{2} \right)}$$

Taking limit $\delta x \rightarrow 0$

$$\begin{aligned} \frac{dy}{dx} &= -2 \sin \left(\frac{x^2 + x^2}{2} \right) \cdot \frac{1}{\frac{2}{2x}} \cdot 1 \\ &= -2 \sin \left(\frac{2x^2}{2} \right) \cdot x \end{aligned}$$

$$\boxed{\frac{d}{dx} (\cos x^2) = -2x \sin x^2} \quad \text{Ans.}$$

(v) $\tan^2 x$

Let

$$\begin{aligned} y &= \tan^2 x \\ y + \delta y &= \tan^2(x + \delta x) \\ \delta y &= \tan^2(x + \delta x) - y \\ \delta y &= \tan^2(x + \delta x) - \tan^2 x \quad \because y = \tan^2 x \\ \delta y &= [\tan(x + \delta x) + \tan x] [\tan(x + \delta x) - \tan x] \\ \delta y &= [\tan(x + \delta x) + \tan x] \left[\frac{\sin(x + \delta x)}{\cos(x + \delta x)} - \frac{\sin x}{\cos x} \right] \\ \delta y &= [\tan(x + \delta x) + \tan x] \left[\frac{\sin(x + \delta x) \cos x - \cos(x + \delta x) \sin x}{\cos(x + \delta x) \cos x} \right] \\ \delta y &= \left[\frac{\tan(x + \delta x) + \tan x}{\cos(x + \delta x) \cdot \cos x} \right] \sin(x + \delta x - x) \end{aligned}$$

Dividing both sides by δx .

$$\frac{\delta y}{\delta x} = \frac{[\tan(x + \delta x) + \tan x]}{\cos(x + \delta x) \cdot \cos x} \cdot \frac{\sin \delta x}{\delta x}$$

Taking limit $\delta x \rightarrow 0$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\tan x + \tan x}{\cos x \cdot \cos x} \cdot 1 \\ \frac{dy}{dx} &= \frac{2 \tan x}{\cos^2 x} \end{aligned}$$

$$\boxed{\frac{d}{dx} (\tan^2 x) = 2 \tan x \sec^2 x} \quad \text{Ans.}$$



(vi) $\sqrt{\tan x}$ (L.B 2003, 2004)

Let

$$y = \sqrt{\tan x}$$

$$y + \delta y = \sqrt{\tan (x + \delta x)}$$

$$\delta y = \sqrt{\tan (x + \delta x)} - y$$

$$\delta y = \sqrt{\tan (x + \delta x)} - \sqrt{\tan x} \quad (\because y = \sqrt{\tan x})$$

$$\delta y = [\sqrt{\tan (x + \delta x)} - \sqrt{\tan x}] \cdot \frac{[\sqrt{\tan (x + \delta x)} + \sqrt{\tan x}]}{[\sqrt{\tan (x + \delta x)} + \sqrt{\tan x}]}$$

$$\delta y = \frac{(\sqrt{\tan (x + \delta x)})^2 - (\sqrt{\tan x})^2}{\sqrt{\tan (x + \delta x)} + \sqrt{\tan x}}$$

$$\delta y = \frac{\tan(x + \delta x) - \tan x}{\sqrt{\tan (x + \delta x)} + \sqrt{\tan x}}$$

$$\delta y = \frac{1}{\sqrt{\tan (x + \delta x)} + \sqrt{\tan x}} \left[\frac{\sin (x + \delta x)}{\cos (x + \delta x)} - \frac{\sin x}{\cos x} \right]$$

$$\delta y = \frac{1}{\sqrt{\tan (x + \delta x)} + \sqrt{\tan x}} \left[\frac{\sin (x + \delta x) \cos x - \cos (x + \delta x) \cdot \sin x}{\cos (x + \delta x) \cdot \cos x} \right]$$

$$\delta y = \frac{1}{\sqrt{\tan (x + \delta x)} + \sqrt{\tan x}} \times \frac{\sin (x + \delta x - x)}{\cos (x + \delta x) \cdot \cos x}$$

$$\delta y = \frac{1}{\sqrt{\tan (x + \delta x)} + \sqrt{\tan x}} \times \frac{\sin \delta x}{\cos (x + \delta x) \cdot \cos x}$$

Dividing both sides by δx

$$\frac{\delta y}{\delta x} = \frac{1}{\sqrt{\tan (x + \delta x)} + \sqrt{\tan x}} \cdot \frac{1}{\cos (x + \delta x) \cdot \cos x} \cdot \frac{\sin \delta x}{\delta x}$$

Taking limit $\delta x \rightarrow 0$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\tan x} + \sqrt{\tan x}} \cdot \frac{1}{\cos x \cdot \cos x} \cdot 1$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\tan x}} \cdot \frac{1}{\cos^2 x}$$

$$\boxed{\frac{d}{dx} (\sqrt{\tan x}) = \frac{\sec^2 x}{2\sqrt{\tan x}}} \quad \text{Ans.}$$

(vii) $\cos \sqrt{x}$ (L.B 2004)

Let

$$\begin{aligned}
 y &= \cos \sqrt{x} \\
 y + \delta y &= \cos \sqrt{x + \delta x} \\
 \delta y &= \cos \sqrt{x + \delta x} - y \\
 \delta y &= \cos \sqrt{x + \delta x} - \cos \sqrt{x} \quad \because y = \cos \sqrt{x} \\
 \delta y &= -2 \sin \left(\frac{\sqrt{x + \delta x} + \sqrt{x}}{2} \right) \cdot \sin \left(\frac{\sqrt{x + \delta x} - \sqrt{x}}{2} \right) \\
 \delta y &= -2 \sin \left(\frac{\sqrt{x + \delta x} + \sqrt{x}}{2} \right) \cdot \frac{\sin \left(\frac{\sqrt{x + \delta x} - \sqrt{x}}{2} \right)}{\delta x} \\
 \delta y &= -2 \sin \left(\frac{\sqrt{x + \delta x} + \sqrt{x}}{2} \right) \cdot \frac{\sin \left(\frac{\sqrt{x + \delta x} - \sqrt{x}}{2} \right)}{2 \left(\sqrt{x + \delta x} + \sqrt{x} \right) \frac{(\sqrt{x + \delta x} - \sqrt{x})}{2}} \\
 &\quad \because \delta x = (\sqrt{x + \delta x} + \sqrt{x}) (\sqrt{x + \delta x} - \sqrt{x})
 \end{aligned}$$

Taking limit $\delta x \rightarrow 0$

$$\frac{dy}{dx} = \frac{-\sin \left(\frac{\sqrt{x} + \sqrt{x}}{2} \right)}{\sqrt{x} + \sqrt{x}} \cdot 1$$

$$\frac{dy}{dx} = \frac{-\sin \left(\frac{2\sqrt{x}}{2} \right)}{2\sqrt{x}}$$

$$\boxed{\frac{d}{dx} (\cos \sqrt{x}) = \frac{-\sin \sqrt{x}}{2\sqrt{x}}} \text{ Ans.}$$

Q.2: Differentiate the following w.r.t. the variable involved.

- (i) $x^2 \sec 4x$ (ii) $\tan^3 \theta \sec^2 \theta$
 (iii) $(\sin 2\theta - \cos 3\theta)^2$ (iv) $\cos \sqrt{x} + \sqrt{\sin x}$

Solution:

- (i) $x^2 \sec 4x$ (G.B 2005)

Let $y = x^2 \sec 4x$

Diff. w.r.t. 'x'



$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx} (x^2 \sec 4x) \\
 &= x^2 \frac{d}{dx} (\sec 4x) + \sec 4x \frac{d}{dx} (x^2) \\
 &= x^2 \sec 4x \tan 4x \cdot 4 + \sec 4x \cdot 2x \\
 &= 2x \sec 4x (1 + 2x \tan 4x) \quad \text{Ans.}
 \end{aligned}$$

(ii) $\tan^3 \theta \sec^2 \theta$

Let $y = \tan^3 \theta \sec^2 \theta$

Diff. w.r.t. 'θ'

$$\begin{aligned}
 \frac{dy}{d\theta} &= \frac{d}{d\theta} (\tan^3 \theta \sec^2 \theta) \\
 &= \tan^3 \theta \frac{d}{d\theta} (\sec^2 \theta) + \sec^2 \theta \frac{d}{d\theta} (\tan^3 \theta) \\
 &= \tan^3 \theta \cdot 2 \sec \theta \cdot \sec \theta \tan \theta + \sec^2 \theta \cdot 3 \tan^2 \theta \cdot \sec^2 \theta \\
 &= \tan^2 \theta \sec^2 \theta (2 \tan^2 \theta + 3 \sec^2 \theta) \quad \text{Ans.}
 \end{aligned}$$

(iii) $(\sin 2\theta - \cos 3\theta)^2$

Let $y = (\sin 2\theta - \cos 3\theta)^2$

Diff. w.r.t. 'θ'

$$\begin{aligned}
 \frac{dy}{d\theta} &= \frac{d}{d\theta} (\sin 2\theta - \cos 3\theta)^2 \\
 \frac{dy}{d\theta} &= 2(\sin 2\theta - \cos 3\theta) \cdot \frac{d}{d\theta} (\sin 2\theta - \cos 3\theta) \\
 \frac{dy}{d\theta} &= 2(\sin 2\theta - \cos 3\theta) (2 \cos 2\theta + 3 \sin 3\theta) \quad \text{Ans.}
 \end{aligned}$$

(iv) $\cos \sqrt{x} + \sqrt{\sin x}$ (L.B 2008)

Let $y = \cos \sqrt{x} + \sqrt{\sin x}$

Diff. w.r.t. 'x'

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx} (\cos \sqrt{x}) + \frac{d}{dx} (\sqrt{\sin x}) \\
 &= -\sin \sqrt{x} \frac{d}{dx} (\sqrt{x}) + \frac{1}{2} (\sin x)^{-\frac{1}{2}} \cdot \cos x \\
 &= \frac{-\sin \sqrt{x}}{2\sqrt{x}} + \frac{\cos x}{2\sqrt{\sin x}} \\
 &= \frac{1}{2} \left(\frac{-\sin \sqrt{x}}{\sqrt{x}} + \frac{\cos x}{\sqrt{\sin x}} \right) \quad \text{Ans.}
 \end{aligned}$$



Q.3: Find $\frac{dy}{dx}$ if

(i) $y = x \cos y$ (L.B 2009) (ii) $x = y \sin y$ (L.B 2009)

Solution:

(i) $y = x \cos y$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx} (x \cos y)$$

$$\frac{dy}{dx} = x \frac{d}{dx} (\cos y) + \cos y \frac{d}{dx} (x)$$

$$\frac{dy}{dx} = x \times (-\sin y \frac{dy}{dx}) + \cos y \cdot 1$$

$$\frac{dy}{dx} + x \sin y \frac{dy}{dx} = \cos y$$

$$\frac{dy}{dx} (1 + x \sin y) = \cos y$$

$$\boxed{\frac{dy}{dx} = \frac{\cos y}{1 + x \sin y}} \quad \text{Ans.}$$

(ii) $x = y \sin y$

Diff. w.r.t. 'x'

$$\frac{d}{dx} (x) = \frac{d}{dx} (y \sin y)$$

$$1 = y \frac{d}{dx} (\sin y) + \sin y \frac{dy}{dx}$$

$$1 = y \cos y \frac{dy}{dx} + \sin y \frac{dy}{dx}$$

$$1 = (y \cos y + \sin y) \frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{y \cos y + \sin y}} \quad \text{Ans.}$$

Q.4: Find the derivative w.r.t. 'x'

(i) $\cos \sqrt{\frac{1+x}{1+2x}}$ (ii) $\sin \sqrt{\frac{1+2x}{1+x}}$

Solution:

(i) $\cos \sqrt{\frac{1+x}{1+2x}}$ (G.B 2005)



Let

$$y = \cos \sqrt{\frac{1+x}{1+2x}}$$

Diff. w.r.t. 'x'

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left(\cos \sqrt{\frac{1+x}{1+2x}} \right) \\ &= -\sin \sqrt{\frac{1+x}{1+2x}} \cdot \frac{d}{dx} \left(\sqrt{\frac{1+x}{1+2x}} \right) \\ &= -\sin \sqrt{\frac{1+x}{1+2x}} \cdot \frac{1}{2} \left(\frac{1+x}{1+2x} \right)^{-\frac{1}{2}} \cdot \frac{d}{dx} \left(\frac{1+x}{1+2x} \right) \\ &= -\sin \sqrt{\frac{1+x}{1+2x}} \cdot \frac{1}{2} \frac{(1+x)^{-\frac{1}{2}}}{(1+2x)^{-\frac{1}{2}}} \left[\frac{(1+2x) \frac{d}{dx} (1+x) - (1+x) \cdot \frac{d}{dx} (1+2x)}{(1+2x)^2} \right] \\ &= -\sin \sqrt{\frac{1+x}{1+2x}} \cdot \frac{1}{2} \frac{[1+2x - (1+x) \cdot 2]}{(1+x)^{\frac{1}{2}} \text{eq } (1+2x)^{2-\frac{1}{2}}} \\ &= -\sin \sqrt{\frac{1+x}{1+2x}} \frac{(1+2x - 2 - 2x)}{2 \sqrt{1+x} (1+2x)^{\frac{3}{2}}} \\ &= -\sin \sqrt{\frac{1+x}{1+2x}} \frac{-1}{2 \sqrt{1+x} (1+2x)^{\frac{3}{2}}} \end{aligned}$$

$\frac{dy}{dx} = \frac{\sin \sqrt{\frac{1+x}{1+2x}}}{2 \sqrt{1+x} (1+2x)^{\frac{3}{2}}}$	Ans.
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(ii) $\sin \sqrt{\frac{1+2x}{1+x}}$ (G.B 2004)

Let $y = \sin \sqrt{\frac{1+2x}{1+x}}$

Diff. w.r.t. 'x'

$$\begin{aligned} \frac{dy}{dx} &= \left[\sin \sqrt{\frac{1+2x}{1+x}} \right] \\ &= \cos \sqrt{\frac{1+2x}{1+x}} \cdot \frac{d}{dx} \left(\sqrt{\frac{1+2x}{1+x}} \right) \end{aligned}$$



$$\begin{aligned}
 &= \cos \sqrt{\frac{1+2x}{1+x}} \cdot \frac{1}{2} \left(\frac{1+2x}{1+x} \right)^{-\frac{1}{2}} \cdot \frac{d}{dx} \left(\frac{1+2x}{1+x} \right) \\
 &= \cos \sqrt{\frac{1+2x}{1+x}} \cdot \frac{1}{2} \frac{(1+2x)^{-\frac{1}{2}}}{(1+x)^{-\frac{1}{2}}} \left[\frac{(1+x) \frac{d}{dx} (1+2x) - (1+2x) \cdot \frac{d}{dx} (1+x)}{(1+x)^2} \right] \\
 &= \cos \sqrt{\frac{1+2x}{1+x}} \cdot \frac{[(1+x)2 - (1+2x)]}{2(1+2x)^{\frac{1}{2}} (1+x)^{\frac{-1}{2}+2}} \\
 &= \cos \sqrt{\frac{1+2x}{1+x}} \cdot \frac{(2+2x-1-2x)}{2\sqrt{1+2x} (1+x)^{\frac{3}{2}}}
 \end{aligned}$$

$$\boxed{\frac{dy}{dx} = \frac{\cos \sqrt{\frac{1+2x}{1+x}}}{2\sqrt{1+2x} (1+x)^{\frac{3}{2}}}}$$

Ans.

Q.5: Differentiate(i) $\sin x$ w.r.t. $\cot x$ (ii) $\sin^2 x$ w.r.t. $\cos^4 x$ **Solution:** $\sin x$ w.r.t. $\cot x$ (**L.B 2009**)

Let

$$y = \sin x, \quad u = \cot x$$

$$y = \sin x$$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \cos x$$

$$u = \cot x$$

Diff. w.r.t. 'x'

$$\frac{du}{dx} = -\operatorname{cosec}^2 x$$

By using chain rule

$$\begin{aligned}
 \frac{dy}{du} &= \frac{dy}{dx} \cdot \frac{dx}{du} \\
 &= \cos x \cdot \frac{-1}{\operatorname{cosec}^2 x}
 \end{aligned}$$

$$\boxed{\frac{dy}{du} = -\cos x \sin^2 x}$$

Ans.

(ii) $\sin^2 x$ w.r.t. $\cos^4 x$ (**G.B 2003, L.B 2009, L.B 2004, L.B 2008**)

Let

$$y = \sin^2 x, \quad u = \cos^4 x$$



$y = \sin^2 x$ Diff. w.r.t. 'x' $\frac{dy}{dx} = 2 \sin x \cos x$	$u = \cos^4 x$ Diff. w.r.t. 'x' $\frac{du}{dx} = 4 \cos^3 x - \sin x$ $= -4 \sin x \cos^3 x$
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By using chain rule

$$\begin{aligned} \frac{dy}{du} &= \frac{dy}{dx} \cdot \frac{dx}{du} \\ &= 2 \sin x \cos x \cdot \frac{-1}{4 \sin x \cos^3 x} \\ &= \frac{-1}{2 \cos^2 x} = \frac{-1}{2} \sec^2 x \quad \text{Ans.} \end{aligned}$$

Q.6: If $\tan y (1 + \tan x) = 1 - \tan x$, show that $\frac{dy}{dx} = -1$ (G.B 2009)

Solution:

$$\tan y (1 + \tan x) = 1 - \tan x$$

$$\tan y = \frac{1 - \tan x}{1 + \tan x}$$

$$\tan y = \frac{\tan \frac{\pi}{4} - \tan x}{1 + \tan \frac{\pi}{4} \tan x}$$

$$\tan y = \tan \left(\frac{\pi}{4} - x \right)$$

$$y = \frac{\pi}{4} - x$$

Diff. w.r.t. 'x'

$$\boxed{\frac{dy}{dx} = -1} \quad \text{Hence proved.}$$

Q.7: If $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \dots \infty}}}$, prove that $(2y - 1) \frac{dy}{dx} = \sec^2 x$.

Solution:

$$y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \dots \infty}}}$$

Squaring on both sides

$$y^2 = \tan x + \sqrt{\tan x + \sqrt{\tan x + \dots \infty}}$$



$$y^2 = \tan x + y$$

Diff. w.r.t. 'x'

$$2y \frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$2y \frac{dy}{dx} - \frac{dy}{dx} = \sec^2 x$$

$$(2y - 1) \frac{dy}{dx} = \sec^2 x$$

Hence proved.

Q.8: If $x = a \cos^3 \theta$, $y = b \sin^3 \theta$, show that $a \frac{dy}{dx} + b \tan \theta = 0$

(G.B 2004, G.B 2011, G.B 2007)

Solution:

$$x = a \cos^3 \theta$$

Diff. w.r.t. 'θ'

$$\begin{aligned} \frac{dx}{d\theta} &= 3a \cos^2 \theta (-\sin \theta) \\ &= -3a \sin \theta \cos^2 \theta \end{aligned}$$

$$y = b \sin^3 \theta$$

Diff. w.r.t. 'θ'

$$\frac{dy}{d\theta} = 3b \sin^2 \theta \cos \theta$$

By using chain rule

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$= 3b \sin^2 \theta \cos \theta \cdot \frac{-1}{3a \sin \theta \cos^2 \theta}$$

$$= \frac{-b}{a} \tan \theta$$

$$a \frac{dy}{dx} = -b \tan \theta$$

$$a \frac{dy}{dx} + b \tan \theta = 0$$

Hence proved.

Q.9: Find $\frac{dy}{dx}$ if $x = a (\cos t + \sin t)$, $y = a (\sin t - t \cos t)$

Solution:

$$x = a (\cos t + \sin t)$$

Diff. w.r.t. 't'

$$\frac{dx}{dt} = a (-\sin t + \cos t)$$

$$y = a (\sin t - t \cos t)$$

Diff. w.r.t. 't'

$$\frac{dy}{dt} = a [\cos t - \{t \cdot -\sin t + \cos t \cdot 1\}]$$



$$\begin{aligned}\frac{dy}{dt} &= a \cos t + a t \cdot \sin t - a \cos t \\ \frac{dy}{dt} &= a t \cdot \sin t\end{aligned}$$

By using chain rule

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{dt}{dx} \\ &= a t \sin t \cdot \frac{1}{a(-\sin t + \cos t)}\end{aligned}$$

$$\boxed{\frac{dy}{dx} = \frac{t \sin t}{\cos t - \sin t}} \text{ Ans.}$$

Q.10: Differentiate w.r.t. 'x'

(i) $\cos^{-1} \frac{x}{a}$

(ii) $\cot^{-1} \frac{x}{a}$

(iii) $\frac{1}{a} \sin^{-1} \frac{a}{x}$

(iv) $\sin^{-1} \sqrt{1-x^2}$

(v) $\sec^{-1} \left(\frac{x^2+1}{x^2-1} \right)$

(vi) $\cot^{-1} \left(\frac{2x}{1-x^2} \right)$

(vii) $\cos^{-1} \left(\frac{1-x^2}{1+x^2} \right)$

Solution:

(i) $\cos^{-1} \frac{x}{a}$

Let $y = \cos^{-1} \frac{x}{a}$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \cdot \frac{d}{dx} \left(\frac{x}{a} \right)$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\frac{x^2}{a^2}}} \cdot \frac{1}{a}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{\frac{a^2-x^2}{a^2}}} \cdot \frac{1}{a}$$



$$\frac{dy}{dx} = \frac{-1}{\sqrt{a^2 - x^2}} \cdot \frac{1}{a}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{a^2 - x^2}} \quad \text{Ans.}$$

(ii) $\cot^{-1} \frac{x}{a}$ (L.B 2006)

Let $y = \cot^{-1} \frac{x}{a}$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{-1}{1 + \left(\frac{x}{a}\right)^2} \cdot \frac{d}{dx} \left(\frac{x}{a}\right)$$

$$\frac{dy}{dx} = \frac{-1}{1 + \frac{x^2}{a^2}} \cdot \frac{1}{a}$$

$$= \frac{-1}{\frac{a^2 + x^2}{a^2}} \cdot \frac{1}{a}$$

$$= \frac{-a}{a^2 + x^2} \quad \text{Ans.}$$

(iii) $\frac{1}{a} \sin^{-1} \frac{a}{x}$ (L.B 2010)

Let $y = \frac{1}{a} \sin^{-1} \frac{a}{x}$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{1}{a} \cdot \frac{1}{\sqrt{1 - \left(\frac{a}{x}\right)^2}} \cdot \frac{d}{dx} \left(\frac{a}{x}\right)$$

$$\frac{dy}{dx} = \frac{1}{a} \cdot \frac{1}{\sqrt{1 - \frac{a^2}{x^2}}} \cdot a \frac{d}{dx} (x^{-1})$$

$$= \frac{1}{\sqrt{\frac{x^2 - a^2}{x^2}}} \cdot -1 x^{-2}$$



$$= \frac{1}{\frac{\sqrt{x^2 - a^2}}{x^2}} \cdot -\frac{1}{x^2}$$

$$= \frac{-1}{x \sqrt{x^2 - a^2}} \text{ Ans.}$$

(iv) $\sin^{-1} \sqrt{1 - x^2}$

Let $y = \sin^{-1} \sqrt{1 - x^2}$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - (\sqrt{1 - x^2})^2}} \cdot \frac{d}{dx} (\sqrt{1 - x^2})$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - 1 + x^2}} \cdot \frac{1}{2} (1 - x^2)^{-\frac{1}{2}} \cdot \frac{d}{dx} (1 - x^2)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{1 - x^2}} \cdot (-2x)$$

$$= \frac{-x}{x \sqrt{1 - x^2}}$$

$$\boxed{\frac{dy}{dx} = \frac{-1}{\sqrt{1 - x^2}}} \text{ Ans.}$$

(v) $\sec^{-1} \left(\frac{x^2 + 1}{x^2 - 1} \right) \quad \left(\because \frac{d}{dx} \sec^{-1} x = \frac{1}{x \sqrt{x^2 - 1}} \right)$

Let

$$y = \sec^{-1} \left(\frac{x^2 + 1}{x^2 - 1} \right)$$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{1}{\left(\frac{x^2 + 1}{x^2 - 1} \right) \sqrt{\left(\frac{x^2 + 1}{x^2 - 1} \right)^2 - 1}} \cdot \frac{d}{dx} \left(\frac{x^2 + 1}{x^2 - 1} \right)$$

$$= \frac{x^2 - 1}{(x^2 + 1) \sqrt{\frac{(x^2 + 1)^2}{(x^2 - 1)^2} - 1}} \left[\frac{(x^2 - 1) \frac{d}{dx} (x^2 + 1) - (x^2 + 1) \frac{d}{dx} (x^2 - 1)}{(x^2 - 1)^2} \right]$$



$$\begin{aligned}
 &= \frac{(x^2-1)2x - (x^2+1) \cdot 2x}{(x^2+1)(x^2-1) \sqrt{\frac{(x^2+1)^2 - (x^2-1)^2}{(x^2-1)^2}}} \\
 &= \frac{2x(x^2-1-x^2-1)}{(x^2+1)(x^2-1) \frac{\sqrt{x^4+1+2x^2 - (x^4+1-2x^2)}}{x^2-1}} \\
 &= \frac{2x(-2)}{(x^2+1) \sqrt{x^4+1+2x^2 - x^4-1+2x^2}} \\
 &= \frac{-4x}{(x^2+1) \sqrt{4x^2}} \\
 &= \frac{-4x}{(x^2+1) \cdot 2x} = \frac{-2}{x^2+1} \quad \text{Ans.}
 \end{aligned}$$

(vi) $\cot^{-1}\left(\frac{2x}{1-x^2}\right)$

Let $y = \cot^{-1}\left(\frac{2x}{1-x^2}\right)$

Diff. w.r.t. 'x' $\left(\frac{d}{dx} \cot^{-1}x = \frac{-1}{1+x^2}\right)$

$$\frac{dy}{dx} = \frac{-1}{1+\left(\frac{2x}{1-x^2}\right)^2} \cdot \frac{d}{dx}\left(\frac{2x}{1-x^2}\right)$$

$$= \frac{-2}{1+\frac{4x^2}{(1-x^2)^2}} \left[\frac{(1-x^2) \frac{d}{dx}(x) - x \frac{d}{dx}(1-x^2)}{(1-x^2)^2} \right]$$

$$= \frac{-2}{\frac{(1-x^2)^2 + 4x^2}{(1-x^2)^2}} \left[\frac{(1-x^2) - x(-2x)}{(1-x^2)^2} \right]$$

$$= \frac{-2 [1-x^2+2x^2]}{1+x^4-2x^2+4x^2}$$

$$= \frac{-2(1+x^2)}{1+x^4+2x^2}$$

$$= \frac{-2(1+x^2)}{(1+x^2)^2}$$



$$= \frac{-2}{1+x^2} \quad \text{Ans.}$$

(vii) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ (L.B 2004)

Let $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}} \cdot \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right) \quad \left(\frac{d}{dx}\cos^{-1}x = \frac{-1}{x\sqrt{1-x^2}}\right)$$

$$= \frac{-1}{\sqrt{1-\frac{(1-x^2)^2}{(1+x^2)^2}}} \left[\frac{(1+x^2)\frac{d}{dx}(1-x^2) - (1-x^2)\frac{d}{dx}(1+x^2)}{(1+x^2)^2} \right]$$

$$= \frac{-1}{\sqrt{\frac{(1+x^2)^2 - (1-x^2)^2}{(1+x^2)^2}}} \left[\frac{(1+x^2)(-2x) - (1-x^2)(2x)}{(1+x^2)^2} \right]$$

$$= \frac{-2x[-1-x^2-1+x^2]}{\frac{\sqrt{1+x^4+2x^2-(1+x^4-2x^2)}}{1+x^2} (1+x^2)^2}$$

$$= \frac{-2x(-2)}{\sqrt{1+x^4+2x^2-1-x^4+2x^2} (1+x^2)}$$

$$= \frac{4x}{\sqrt{4x^2} (1+x^2)}$$

$$= \frac{4x}{2x(1+x^2)}$$

$$= \frac{2}{1+x^2} \quad \text{Ans.}$$

Q.11: Show that $\frac{dy}{dx} = \frac{y}{x}$ if $\frac{y}{x} = \tan^{-1} \frac{x}{y}$ (L.B 2003, G.B 2007, L.B 2007, G.B 2008)

Solution:



$$\frac{y}{x} = \tan^{-1} \frac{x}{y}$$

$$y = x \tan^{-1} \frac{x}{y}$$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = x \frac{d}{dx} \left(\tan^{-1} \frac{x}{y} \right) + \tan^{-1} \left(\frac{x}{y} \right) \frac{d}{dx} (x)$$

$$\frac{dy}{dx} = x \cdot \frac{1}{1 + \left(\frac{x}{y} \right)^2} \cdot \frac{d}{dx} \left(\frac{x}{y} \right) + \tan^{-1} \left(\frac{x}{y} \right)$$

$$\frac{dy}{dx} = \frac{x}{1 + \frac{x^2}{y^2}} \left[\frac{y \frac{d}{dx} (x) - x \frac{dy}{dx}}{y^2} \right] + \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{x}{1 + \frac{x^2}{y^2}} \left[\frac{y - x \frac{dy}{dx}}{y^2} \right] + \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{x}{\frac{y^2 + x^2}{y^2}} \left[\frac{\left(y - \frac{dy}{dx} \right)}{y^2} \right] + \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{xy}{x^2 + y^2} - \frac{x^2}{x^2 + y^2} \frac{dy}{dx} + \frac{y}{x}$$

$$\frac{dy}{dx} + \frac{x^2}{x^2 + y^2} \frac{dy}{dx} = y \left[\frac{x}{x^2 + y^2} + \frac{1}{x} \right]$$

$$\frac{dy}{dx} \left[1 + \frac{x^2}{x^2 + y^2} \right] = y \left[\frac{x^2 + x^2 + y^2}{x(x^2 + y^2)} \right]$$

$$\frac{dy}{dx} \left(\frac{x^2 + y^2 + x^2}{x^2 + y^2} \right) = y \left[\frac{2x^2 + y^2}{x(x^2 + y^2)} \right] = \frac{y(x^2 + y^2)(2x^2 + y^2)}{x(x^2 + y^2)(2x^2 + y^2)}$$

$$\boxed{\frac{dy}{dx} = \frac{y}{x}}$$

Hence proved.

Q.12: If $y = \tan (P \tan^{-1} x)$, show that $(1 + x^2) y_1 - P (1 + y^2) = 0$

(L.B 2006) (G.B 2006)



Solution:

$$\begin{aligned} y &= \tan (P \tan^{-1} x) \\ \tan^{-1} y &= P \tan^{-1} x \end{aligned}$$

Diff. w.r.t. 'x'

$$\frac{1}{1+y^2} \cdot y_1 = P \frac{1}{1+x^2}$$

$$(1+x^2) y_1 = P (1+y^2)$$

$$(1+x^2) y_1 - P (1+y^2) = 0$$

Hence proved.

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EXERCISE 2.6

Q.1: Find $f'(x)$ if

(i) $f(x) = e^{\sqrt{x}-1}$

(ii) $f(x) = x^3 e^{\frac{1}{x}} (x \neq 0)$

(iii) $f(x) = ex(1 + \ln x)$

(iv) $f(x) = \frac{e^x}{e^{-x} + 1}$

(v) $f(x) = \ln(e^x + e^{-x})$

(vi) $f(x) = \frac{e^{ax} - e^{-ax}}{e^{ax} + e^{-ax}}$

(vii) $f(x) = \sqrt{\ln(e^{2x} + e^{-2x})}$

(viii) $f(x) = \ln(\sqrt{e^{2x} + e^{-2x}})$

Solution:

(i) $f(x) = e^{\sqrt{x}-1}$

Diff. w.r.t. 'x'

$$f'(x) = e^{\sqrt{x}-1} \cdot \frac{d}{dx}(\sqrt{x}-1)$$

$$f'(x) = e^{\sqrt{x}-1} \cdot \frac{1}{2} x^{-\frac{1}{2}}$$

$$f'(x) = \frac{e^{\sqrt{x}-1}}{2\sqrt{x}}$$

Ans.

(ii) $f(x) = x^3 e^{\frac{1}{x}} (x \neq 0)$

Diff. w.r.t. 'x'

$$f'(x) = x^3 \frac{d}{dx} \left(e^{\frac{1}{x}} \right) + e^{\frac{1}{x}} \frac{d}{dx} (x^3)$$



$$f(x) = x^3 e^{\frac{1}{x}} \cdot \frac{d}{dx} \left(\frac{1}{x} \right) + e^{\frac{1}{x}} 3x^2$$

$$f(x) = x^3 e^{\frac{1}{x}} \times \frac{-1}{x^2} + 3x^2 e^{\frac{1}{x}}$$

$$f(x) = -x e^{\frac{1}{x}} + 3x^2 e^{\frac{1}{x}}$$

$$\boxed{f(x) = x e^{\frac{1}{x}} (3x - 1)} \quad \text{Ans.}$$

(iii) $f(x) = e^x (1 + \ln x)$

Diff. w.r.t. 'x'

$$f(x) = e^x \frac{d}{dx} (1 + \ln x) + (1 + \ln x) \frac{d}{dx} (e^x)$$

$$f(x) = e^x \cdot \frac{1}{x} + (1 + \ln x) e^x$$

$$f(x) = e^x \left[\frac{1}{x} + (1 + \ln x) \right]$$

$$\boxed{f(x) = \frac{e^x [1 + x(1 + \ln x)]}{x}} \quad \text{Ans}$$

(iv) $f(x) = \frac{e^x}{e^{-x} + 1}$

Diff. w.r.t. 'x'

$$f(x) = \frac{(e^{-x} + 1) \frac{d}{dx} (e^x) - e^x \frac{d}{dx} (e^{-x} + 1)}{(e^{-x} + 1)^2}$$

$$f(x) = \frac{(e^{-x} + 1)e^x - e^x \cdot e^{-x}(-1)}{(e^{-x} + 1)^2}$$

$$f(x) = \frac{e^{-x} \cdot e^x + e^x + e^{x-x}}{(e^{-x} + 1)^2}$$

$$f(x) = \frac{e^{-x+x} + e^x + e^0}{(e^{-x} + 1)^2}$$

$$f(x) = \frac{e^0 + e^x + 1}{(e^{-x} + 1)^2}$$

$$f(x) = \frac{1 + e^x + 1}{(e^{-x} + 1)^2}$$



$$f(x) = \frac{e^x + 2}{(e^{-x} + 1)^2}$$

Ans.

(v) $f(x) = \ln(e^x + e^{-x})$

Diff. w.r.t. 'x'

$$f'(x) = \frac{1}{e^x + e^{-x}} \frac{d}{dx} (e^x + e^{-x})$$

$$f'(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$f'(x) = \frac{e^x - \frac{1}{e^x}}{e^x + \frac{1}{e^x}}$$

$$f'(x) = \frac{\frac{e^{2x} - 1}{e^x}}{\frac{e^{2x} + 1}{e^x}}$$

$$f'(x) = \frac{e^{2x} - 1}{e^{2x} + 1}$$

Ans.

(vi) $f(x) = \frac{e^{ax} - e^{-ax}}{e^{ax} + e^{-ax}}$

Diff. w.r.t. 'x'

$$f'(x) = \frac{(e^{ax} + e^{-ax}) \frac{d}{dx} (e^{ax} - e^{-ax}) - (e^{ax} - e^{-ax}) \frac{d}{dx} (e^{ax} + e^{-ax})}{(e^{ax} + e^{-ax})^2}$$

$$f'(x) = \frac{(e^{ax} + e^{-ax}) (e^{ax} \cdot a - e^{-ax} (-a)) - (e^{ax} - e^{-ax}) (e^{ax} \cdot a + e^{-ax} \times -a)}{(e^{ax} + e^{-ax})^2}$$

$$f'(x) = \frac{a(e^{ax} + e^{-ax}) (e^{ax} + e^{-ax}) - a(e^{ax} - e^{-ax}) (e^{ax} - e^{-ax})}{(e^{ax} + e^{-ax})^2}$$

$$f'(x) = \frac{a[(e^{ax} + e^{-ax})^2 - (e^{ax} - e^{-ax})^2]}{(e^{ax} + e^{-ax})^2}$$

$$f'(x) = \frac{a[e^{2ax} + e^{-2ax} + 2e^{ax}e^{-ax} - (e^{2ax} + e^{-2ax} - 2e^{ax} \cdot e^{-ax})]}{(e^{ax} + e^{-ax})^2}$$

$$f'(x) = \frac{a(e^{2ax} + e^{-2ax} + 2 - e^{2ax} - e^{-2ax} + 2)}{(e^{ax} + e^{-ax})^2}$$



$$f'(x) = \frac{4a}{(e^{ax} + e^{-ax})^2} \quad \text{Ans.}$$

(vii) $f(x) = \sqrt{\ln(e^{2x} + e^{-2x})}$

Diff. w.r.t. 'x'

$$f'(x) = \frac{1}{2} [\ln(e^{2x} + e^{-2x})]^{-1/2} \cdot \frac{d}{dx} [\ln(e^{2x} + e^{-2x})]$$

$$f'(x) = \frac{1}{2\sqrt{\ln(e^{2x} + e^{-2x})}} \cdot \frac{1}{e^{2x} + e^{-2x}} \cdot \frac{d}{dx} (e^{2x} + e^{-2x})$$

$$f'(x) = \frac{1}{2\sqrt{\ln(e^{2x} + e^{-2x})}} \cdot \frac{1}{e^{2x} + e^{-2x}} \cdot (e^{2x} \cdot 2 + e^{-2x}(-2))$$

$$f'(x) = \frac{1}{2\sqrt{\ln(e^{2x} + e^{-2x})}} \cdot \frac{1}{e^{2x} + e^{-2x}} \cdot 2(e^{2x} - e^{-2x})$$

$$f'(x) = \frac{e^{2x} - e^{-2x}}{(e^{2x} + e^{-2x})\sqrt{\ln(e^{2x} + e^{-2x})}} \quad \text{Ans.}$$

(viii) $f(x) = \ln(\sqrt{e^{2x} + e^{-2x}}) \quad (L.B 2009 (s))$

Diff. w.r.t. 'x'

$$f'(x) = \frac{1}{\sqrt{e^{2x} + e^{-2x}}} \cdot \frac{d}{dx} (\sqrt{e^{2x} + e^{-2x}})$$

$$f'(x) = \frac{1}{\sqrt{e^{2x} + e^{-2x}}} \cdot \frac{1}{2} (e^{2x} + e^{-2x})^{-1/2} \cdot \frac{d}{dx} (e^{2x} + e^{-2x})$$

$$f'(x) = \frac{1}{2\sqrt{e^{2x} + e^{-2x}} \sqrt{e^{2x} + e^{-2x}}} (e^{2x} \cdot 2 + e^{-2x}(-2))$$

$$f'(x) = \frac{2(e^{2x} - e^{-2x})}{2(e^{2x} + e^{-2x})} = \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}$$

$$f'(x) = \tanh 2x \quad \text{Ans.}$$

Q.2: Find $\frac{dy}{dx}$ if

(i) $y = x^2 \ln \sqrt{x}$

(ii) $y = x \sqrt{\ln x}$

(iii) $y = \frac{x}{\ln x}$

(iv) $y = x^2 \ln \frac{1}{x}$



$$(v) \quad y = \ell n \sqrt{\frac{x^2-1}{x^2+1}}$$

$$(vi) \quad y = \ell n (x + \sqrt{x^2+1})$$

$$(vii) \quad y = \ell n (9 - x^2) \quad (L.B \ 2009)$$

$$(viii) \quad y = e^{-2x} \sin 2x \quad (L.B \ 2009 \ (s))$$

$$(ix) \quad y = e^{-x} (x^3 + 2x^2 + 1) \quad (L.B \ 2009) \quad (x)$$

$$y = x e^{\sin x}$$

$$(xi) \quad y = 5e^{3x-4}$$

$$(xii) \quad y = (x+1)^x$$

$$(xiii) \quad y = (\ell n x)^{\ell n x}$$

$$(xiv) \quad y = \frac{\sqrt{x^2-1} (x+1)}{(x^3+1)^{\frac{3}{2}}}$$

Solution:

$$(i) \quad y = x^2 \ell n \sqrt{x}$$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = x^2 \frac{d}{dx} (\ell n \sqrt{x}) + \ell n \sqrt{x} \frac{d}{dx} (x^2)$$

$$\frac{dy}{dx} = x^2 \cdot \frac{1}{\sqrt{x}} \cdot \frac{d}{dx} (\sqrt{x}) + \ell n \sqrt{x} \cdot 2x$$

$$\frac{dy}{dx} = \frac{x^2}{\sqrt{x}} \cdot \frac{1}{2} x^{-\frac{1}{2}} + 2x \ell n \sqrt{x}$$

$$\frac{dy}{dx} = \frac{x^2}{2\sqrt{x} \cdot \sqrt{x}} + 2x \ell n \sqrt{x}$$

$$\frac{dy}{dx} = \frac{x^2}{2x} + 2x \ell n \sqrt{x}$$

$$\boxed{\frac{dy}{dx} = \frac{x}{2} + 2x \ell n \sqrt{x}} \quad \text{Ans.}$$

$$(ii) \quad y = x \sqrt{\ell n x}$$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = x \frac{d}{dx} (\sqrt{\ell n x}) + \sqrt{\ell n x} \frac{d}{dx} (x)$$

$$\frac{dy}{dx} = x \cdot \frac{1}{2} (\ell n x)^{-\frac{1}{2}} \cdot \frac{d}{dx} (\ell n x) + \sqrt{\ell n x} \cdot 1$$

$$\frac{dy}{dx} = \frac{x}{2\sqrt{\ell n x}} \cdot \frac{1}{x} + \sqrt{\ell n x}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\ell n x}} + \sqrt{\ell n x}$$



$$\boxed{\frac{dy}{dx} = \frac{1 + 2\ln x}{2\sqrt{\ln x}}} \quad \text{Ans}$$

(iii) $y = \frac{x}{\ln x}$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{(\ln x) \frac{d}{dx}(x) - x \frac{d}{dx}(\ln x)}{(\ln x)^2}$$

$$\frac{dy}{dx} = \frac{\ln x \cdot 1 - x \cdot \frac{1}{x}}{(\ln x)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{\ln x - 1}{(\ln x)^2}} \quad \text{Ans.}$$

(iv) $y = x^2 \ln \frac{1}{x}$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = x^2 \frac{d}{dx} \left(\ln \frac{1}{x} \right) + \ln \frac{1}{x} \frac{d}{dx} (x^2)$$

$$\frac{dy}{dx} = x^2 \cdot \frac{1}{\frac{1}{x}} \cdot \frac{d}{dx} \left(\frac{1}{x} \right) + \ln \frac{1}{x} \cdot 2x$$

$$\frac{dy}{dx} = x^3 \cdot \frac{-1}{x^2} + 2x \ln \frac{1}{x}$$

$$\frac{dy}{dx} = -x + 2x \ln \frac{1}{x}$$

$$\boxed{\frac{dy}{dx} = x \left[2 \ln \frac{1}{x} - 1 \right]} \quad \text{Ans.}$$

(v) $y = \ln \sqrt{\frac{x^2 - 1}{x^2 + 1}}$

$$y = \frac{1}{2} \ln \left(\frac{x^2 - 1}{x^2 + 1} \right)$$

Diff. w.r.t. 'x'



$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{\frac{x^2-1}{x^2+1}} \cdot \frac{d}{dx} \left(\frac{x^2-1}{x^2+1} \right)$$

$$\frac{dy}{dx} = \frac{x^2+1}{2(x^2-1)} \left[\frac{(x^2+1) \frac{d}{dx} (x^2-1) - (x^2-1) \frac{d}{dx} (x^2+1)}{(x^2+1)^2} \right]$$

$$\frac{dy}{dx} = \frac{(x^2+1)2x - (x^2-1)2x}{2(x^2-1)(x^2+1)}$$

$$\frac{dy}{dx} = \frac{2x(x^2+1 - x^2+1)}{2(x^4-1)}$$

$$\boxed{\frac{dy}{dx} = \frac{2x}{x^4-1}} \quad \text{Ans.}$$

(vi) $y = \ln(x + \sqrt{x^2+1})$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{1}{x + \sqrt{x^2+1}} \cdot \frac{d}{dx} (x + \sqrt{x^2+1})$$

$$\frac{dy}{dx} = \frac{1}{x + \sqrt{x^2+1}} \cdot \left[1 + \frac{1}{2} (x^2+1)^{-\frac{1}{2}} \cdot \frac{d}{dx} (x^2+1) \right]$$

$$\frac{dy}{dx} = \frac{1}{x + \sqrt{x^2+1}} \cdot \left[1 + \frac{1}{2\sqrt{x^2+1}} \cdot 2x \right]$$

$$\frac{dy}{dx} = \frac{1}{x + \sqrt{x^2+1}} \cdot \left[\frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1}} \right]$$

$$\boxed{\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}} \quad \text{Ans.}$$

(vii) $y = \ln(9 - x^2)$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{1}{9-x^2} \cdot \frac{d}{dx} (9-x^2)$$

$$\boxed{\frac{dy}{dx} = \frac{-2x}{9-x^2}} \quad \text{Ans.}$$

(viii) $y = e^{-2x} \sin 2x$

Diff. w.r.t. 'x'



$$\frac{dy}{dx} = e^{-2x} \frac{d}{dx} (\sin 2x) + \sin 2x \frac{d}{dx} (e^{-2x})$$

$$\frac{dy}{dx} = e^{-2x} \cdot \cos 2x (2) + \sin 2x \cdot e^{-2x} \cdot (-2)$$

$$\boxed{\frac{dy}{dx} = 2e^{-2x} (\cos 2x - \sin 2x)} \quad \text{Ans.}$$

(ix) $y = e^{-x} (x^3 + 2x^2 + 1)$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = e^{-x} \frac{d}{dx} (x^3 + 2x^2 + 1) + (x^3 + 2x^2 + 1) \frac{d}{dx} (e^{-x})$$

$$\frac{dy}{dx} = e^{-x} (3x^2 + 4x) + (x^3 + 2x^2 + 1) \cdot e^{-x} (-1)$$

$$\frac{dy}{dx} = e^{-x} (3x^2 + 4x - x^3 - 2x^2 - 1)$$

$$\frac{dy}{dx} = e^{-x} (-x^3 + x^2 + 4x - 1)$$

$$\boxed{\frac{dy}{dx} = -e^{-x} (x^3 - x^2 - 4x + 1)} \quad \text{Ans.}$$

(x) $y = x e^{\sin x}$ (G.B 2007)

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = x \frac{d}{dx} (e^{\sin x}) + e^{\sin x} \frac{d}{dx} (x)$$

$$\frac{dy}{dx} = x e^{\sin x} \frac{d}{dx} (\sin x) + e^{\sin x} \cdot 1$$

$$\frac{dy}{dx} = x e^{\sin x} \cos x + e^{\sin x}$$

$$\boxed{\frac{dy}{dx} = e^{\sin x} (x \cos x + 1)} \quad \text{Ans.}$$

(xi) $y = 5e^{3x-4}$ (G.B 2006)

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = 5 e^{3x-4} \cdot \frac{d}{dx} (3x - 4)$$

$$\frac{dy}{dx} = 5 e^{3x-4} \cdot 3$$



$$\boxed{\frac{dy}{dx} = 15 e^{3x-4}}$$

Ans.

(xii) $y = (x+1)^x$

Taking 'ln' on both sides

$$\ln y = \ln (x+1)^x$$

$$\ln y = x \ln (x+1)$$

Diff. w.r.t. 'x'

$$\frac{1}{y} \cdot \frac{dy}{dx} = x \frac{d}{dx} [\ln (x+1)] + \ln (x+1) \frac{d}{dx} (x)$$

$$\frac{dy}{dx} = y \left[x \cdot \frac{1}{x+1} + \ln (x+1) \right]$$

$$\boxed{\frac{dy}{dx} = (x+1)^x \left[\frac{x + (x+1) \ln (x+1)}{x+1} \right]}$$

Ans.

(xiii) $y = (\ln x)^{\ln x}$

Taking 'ln' on both sides

$$\ln y = \ln (\ln x)^{\ln x}$$

$$\ln y = \ln x \ln (\ln x)$$

Diff. w.r.t. 'x'

$$\frac{1}{y} \frac{dy}{dx} = \ln x \frac{d}{dx} [\ln (\ln x)] + \ln (\ln x) \frac{d}{dx} (\ln x)$$

$$\frac{dy}{dx} = y \left[\ln x \cdot \frac{1}{\ln x} \cdot \frac{d}{dx} (\ln x) + \ln (\ln x) \cdot \frac{1}{x} \right]$$

$$\frac{dy}{dx} = (\ln x)^{\ln x} \left[\frac{1}{x} + \frac{\ln (\ln x)}{x} \right]$$

$$\boxed{\frac{dy}{dx} = (\ln x)^{\ln x} \left[1 + \frac{\ln (\ln x)}{x} \right]}$$

Ans.

(xiv) $y = \frac{\sqrt{x^2-1} (x+1)}{(x^3+1)^{\frac{3}{2}}}$

$$y = \frac{\sqrt{(x+1)(x-1)} (x+1)}{[(x+1)(x^2-x+1)]^{\frac{3}{2}}}$$



$$y = \frac{\sqrt{x+1} \cdot \sqrt{x-1} (x+1)}{(x+1)^{\frac{3}{2}} (x^2-x+1)^{\frac{3}{2}}}$$

$$y = \frac{(x-1)^{\frac{1}{2}}}{(x^2-x+1)^{\frac{3}{2}}}$$

Taking 'ln' on both sides

$$\ell n y = \ell n \left[\frac{(x-1)^{\frac{1}{2}}}{(x^2-x+1)^{\frac{3}{2}}} \right]$$

$$\ell n y = \ell n (x-1)^{\frac{1}{2}} - \ell n (x^2-x+1)^{\frac{3}{2}}$$

$$\ell n y = \frac{1}{2} \ell n (x-1) - \frac{3}{2} \ell n (x^2-x+1)$$

Diff. w.r.t. 'x'

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{x-1} \cdot \frac{d}{dx} (x-1) - \frac{3}{2} \cdot \frac{1}{x^2-x+1} \cdot \frac{d}{dx} (x^2-x+1)$$

$$\frac{dy}{dx} = y \left[\frac{1}{2(x-1)} - \frac{3(2x-1)}{2(x^2-x+1)} \right]$$

$$\frac{dy}{dx} = \frac{(x-1)^{\frac{1}{2}}}{(x^2-x+1)^{\frac{3}{2}}} \left[\frac{x^2-x+1-3(2x-1)(x-1)}{2(x-1)(x^2-x+1)} \right]$$

$$\frac{dy}{dx} = \frac{x^2-x+1-3(2x^2-3x+1)}{2\sqrt{x-1} (x^2-x+1)^{\frac{3}{2}+1}}$$

$$\frac{dy}{dx} = \frac{x^2-x+1-6x^2+9x-3}{2\sqrt{x-1} (x^2-x+1)^{\frac{5}{2}}}$$

$$\frac{dy}{dx} = \frac{-5x^2+8x-2}{2\sqrt{x-1} (x^2-x+1)^{\frac{5}{2}}}$$

$$\boxed{\frac{dy}{dx} = -\frac{5x^2-8x+2}{2\sqrt{x-1} (x^2-x+1)^{\frac{5}{2}}}}$$

Ans.

Q.3: Find $\frac{dy}{dx}$ if



- (i) $y = \cosh 2x$ (ii) $y = \sinh 3x$
 (iii) $y = \tanh^{-1}(\sin x) - \frac{\pi}{2} < x < \frac{\pi}{2}$
 (iv) $y = \sinh^{-1}(x^3)$ (v) $y = (\ln \tan hx)$
 (vi) $y = \sinh^{-1}\left(\frac{x}{2}\right)$

Solution:

- (i) $y = \cosh 2x$ (L.B 2008)

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \sinh 2x \cdot \frac{d}{dx}(2x)$$

$$\boxed{\frac{dy}{dx} = 2 \sinh 2x} \quad \text{Ans.}$$

- (ii) $y = \sinh 3x$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \cosh 3x \cdot \frac{d}{dx}(3x)$$

$$\boxed{\frac{dy}{dx} = 3 \cosh 3x} \quad \text{Ans.}$$

- (iii) $y = \tanh^{-1}(\sin x) \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{1}{1 - \sin^2 x} \cdot \frac{d}{dx}(\sin x)$$

$$\boxed{\frac{dy}{dx} = \frac{\cos x}{\cos^2 x} = \frac{1}{\cos x} = \sec x} \quad \text{Ans.}$$

- (iv) $y = \sinh^{-1}(x^3)$ (L.B 2008), (G.B 2008)

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 + (x^3)^2}} \cdot \frac{d}{dx}(x^3)$$

$$\boxed{\frac{dy}{dx} = \frac{3x^2}{\sqrt{1 + x^6}}} \quad \text{Ans.}$$

- (v) $y = (\ln \tan hx)$ (L.B 2009 (s))



Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{1}{\tanh x} \cdot \frac{d}{dx} (\tanh x)$$

$$\frac{dy}{dx} = \frac{1}{\tanh x} \cdot \sec^2 x$$

$$\frac{dy}{dx} = \frac{\frac{1}{\cosh^2 x}}{\frac{\sinh x}{\cosh x}}$$

$$\frac{dy}{dx} = \frac{1}{\sinh x \cosh x}$$

$$\frac{dy}{dx} = \frac{2}{2 \sinh x \cosh x} = \frac{2}{\sinh 2x}$$

$$\boxed{\frac{dy}{dx} = 2 \operatorname{cosech} 2x}$$

Ans.

(vi) $y = \sinh^{-1} \left(\frac{x}{2} \right)$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 + \left(\frac{x}{2}\right)^2}} \cdot \frac{d}{dx} \left(\frac{x}{2} \right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 + \frac{x^2}{4}}} \cdot \frac{1}{2}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\frac{4 + x^2}{4}}} \cdot \frac{1}{2}$$

$$\frac{dy}{dx} = \frac{1}{\frac{\sqrt{4 + x^2}}{2}} \cdot \frac{1}{2}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{\sqrt{4 + x^2}}}$$

Ans.



EXERCISE 2.7

Q.1: Find y_2 if

(i) $y = 2x^5 - 3x^4 + 4x^3 + x - 2$ (ii) $y = (2x + 5)^{\frac{3}{2}}$

(iii) $y = \sqrt{x} + \frac{1}{\sqrt{x}}$

Solution:

(i) $y = 2x^5 - 3x^4 + 4x^3 + x - 2$

$$y_1 = 10x^4 - 12x^3 + 12x^2 + 1$$

$$y_2 = 40x^3 - 36x^2 + 24x$$

Ans.

(ii) $y = (2x + 5)^{\frac{3}{2}}$

$$y_1 = \frac{3}{2} (2x + 5)^{\frac{3}{2} - 1} \cdot \frac{d}{dx} (2x + 5)$$

$$y_1 = \frac{3}{2} (2x + 5)^{\frac{1}{2} - 1} \cdot 2$$

$$y_1 = 3(2x + 5)^{\frac{1}{2}}$$

$$y_2 = \frac{3}{2} (2x + 5)^{\frac{1}{2} - 1} \cdot \frac{d}{dx} (2x + 5)$$

$$y_2 = \frac{3}{2} (2x + 5)^{-\frac{1}{2}} \cdot 2$$

$$y_2 = \frac{3}{\sqrt{2x + 5}}$$

Ans.

(iii) $y = \sqrt{x} + \frac{1}{\sqrt{x}}$

$$y = x^{\frac{1}{2}} + x^{-\frac{1}{2}}$$

Diff. w.r.t. 'x'

$$y_1 = \frac{1}{2} x^{-\frac{1}{2}} + \left(-\frac{1}{2} \right) x^{-\frac{3}{2}}$$

$$y_1 = \frac{1}{2} x^{-\frac{1}{2}} - \frac{1}{2} x^{-\frac{3}{2}}$$

$$y_2 = \frac{1}{2} \left(-\frac{1}{2} \right) x^{-\frac{3}{2}} - \left(-\frac{1}{2} \right) \left(-\frac{3}{2} \right) x^{-\frac{5}{2}}$$



$$y_2 = \frac{-1}{\frac{3}{4x^2}} + \frac{3}{\frac{5}{4x^2}}$$

$$y_2 = \frac{-x+3}{\frac{5}{4x^2}}$$

Ans.

Q.2: Find y_2 if

(i) $y = x^2 \cdot e^{-x}$ (L.B 2011) (L.B 2008)

(ii) $y = \ln\left(\frac{2x+3}{3x+2}\right)$ (L.B 2009)

Solution:

(i) $y = x^2 \cdot e^{-x}$

$$y_1 = x^2 \frac{d}{dx} (e^{-x}) + e^{-x} \frac{d}{dx} (x^2)$$

$$y_1 = x^2 \cdot e^{-x} (-1) + e^{-x} 2x$$

$$y_1 = -x^2 e^{-x} + 2x e^{-x}$$

$$y_1 = e^{-x} (2x - x^2)$$

$$y_2 = e^{-x} \frac{d}{dx} (2x - x^2) + (2x - x^2) \cdot \frac{d}{dx} (e^{-x})$$

$$y_2 = e^{-x} (2 - 2x) + (2x - x^2) \cdot e^{-x} \cdot (-1)$$

$$y_2 = e^{-x} (2 - 2x - 2x + x^2)$$

$$y_2 = e^{-x} (x^2 - 4x + 2)$$

Ans.

(ii) $y = \ln\left(\frac{2x+3}{3x+2}\right)$

$$y = \ln(2x+3) - \ln(3x+2)$$

$$y_1 = \frac{1}{2x+3} \cdot 2 - \frac{1}{3x+2} \cdot 3$$

$$y_1 = 2(2x+3)^{-1} - 3(3x+2)^{-1}$$

$$y_2 = 2(-1)(2x+3)^{-2} \cdot 2 - 3(-1)(3x+2)^{-2} \cdot 3$$

$$y_2 = \frac{-4}{(2x+3)^2} + \frac{9}{(3x+2)^2}$$

$$y_2 = \frac{-4(3x+2)^2 + 9(2x+3)^2}{(2x+3)^2(3x+2)^2}$$

$$y_2 = \frac{-4(9x^2 + 4 + 12x)^2 + 9(4x^2 + 9 + 12x)}{(2x+3)^2(3x+2)^2}$$



$$y_2 = \frac{-36x^2 - 16 - 48x + 36x^2 + 81 + 108x}{(2x+3)^2(3x+2)^2}$$

$$y_2 = \frac{60x + 65}{(2x+3)^2(3x+2)^2}$$

Ans.

Q.3: Find y_2 if

(i) $x^2 + y^2 = a^2$ (ii) $x^3 - y^3 = a^3$

(iii) $x = a \cos \theta$, $y = a \sin \theta$ (G.B 2006)

(iv) $x = at^2$, $y = bt^4$

(v) $x^2 + y^2 + 2gx + 2fy + c = 0$

Solution:

(i) $x^2 + y^2 = a^2$

$$2x + 2yy_1 = 0$$

$$2yy_1 = -2x$$

$$y_1 = \frac{-2x}{2y}$$

$$y_1 = \frac{-x}{y}$$

$$y_2 = -\frac{y \cdot 1 - x \cdot y_1}{y^2}$$

$$y_2 = -\frac{y - x \left(\frac{-x}{y} \right)}{y^2}$$

$$y_2 = -\frac{y + \frac{x^2}{y}}{y^2}$$

$$= -\frac{\frac{y^2 + x^2}{y}}{y^2}$$

$$y_2 = -\frac{y^2 + x^2}{y^3}$$

$$y_2 = \frac{-a^2}{y^3}$$

$$\because x^2 + y^2 = a^2$$

Ans.

(ii) $x^3 - y^3 = a^3$

$$3x^2 - 3y^2y_1 = 0$$

$$-3y^2y_1 = -3x^2$$



$$y_1 = \frac{-3x^2}{-3y^2}$$

$$y_1 = \frac{x^2}{y^2}$$

$$y_2 = \frac{y^2 \frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(y^2)}{(y^2)^2}$$

$$y_2 = \frac{y^2 \cdot 2x - x^2 \cdot 2yy_1}{y^4}$$

$$y_2 = \frac{2xy^2 - 2x^2y \left(\frac{x^2}{y^2}\right)}{y^4}$$

$$y_2 = \frac{2xy^2 - \frac{2x^4}{y}}{y^4}$$

$$y_2 = \frac{2xy^3 - 2x^4}{y^4}$$

$$y_2 = \frac{-2x(x^3 - y^3)}{y^5}$$

$$\boxed{y_2 = \frac{-2x a^3}{y^5}}$$

Ans.

$$\because x^3 - y^3 = a^3$$

(iii) $x = a \cos \theta$, $y = a \sin \theta$

$$x = a \cos \theta$$

Diff. w.r.t. 'θ'

$$\frac{dx}{d\theta} = -a \sin \theta$$

$$y = a \sin \theta$$

Diff. w.r.t. 'θ'

$$\frac{dy}{d\theta} = a \cos \theta$$

By using chain rule

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$y_1 = a \cos \theta \cdot \frac{-1}{a \sin \theta}$$

$$y_1 = -\cot \theta$$

$$y_2 = -(-\operatorname{cosec}^2 \theta) \cdot \frac{-1}{a \sin \theta}$$



$$y_2 = \frac{-1}{a \sin^3 \theta} \quad \text{Ans.}$$

(iv) $x = at^2, \quad y = bt^4$

$$x = at^2$$

Diff. w.r.t. 't'

$$\frac{dx}{dt} = 2at$$

$$y = bt^4$$

Diff. w.r.t. 't'

$$\frac{dy}{dt} = 4bt^3$$

By using chain rule

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$y_1 = 4bt^3 \cdot \frac{1}{2at}$$

$$y_1 = \frac{2b}{a} t^2$$

$$y_2 = \frac{x}{a} (2t) \frac{dt}{dx} \Rightarrow y_2 = \frac{4bt}{a} \cdot \frac{1}{2at}$$

$$y_2 = \frac{2b}{a^2} \quad \text{Ans.}$$

(v) $x^2 + y^2 + 2gx + 2fy + c = 0$

$$2x^2 + 2yy_1 + 2g + 2fy_1 = 0$$

$$2(y+f)y_1 = -2x-2g$$

$$y_1 = \frac{-2(x+g)}{2(y+f)}$$

$$y_1 = -\frac{x+g}{y+f}$$

$$y_2 = -\frac{(y+f) \frac{d}{dx}(x+g) - (x+g) \frac{d}{dx}(y+f)}{(y+f)^2}$$

$$y_2 = -\frac{(y+f) - (x+g) \cdot y_1}{(y+f)^2}$$

$$y_2 = -\frac{(y+f) - (x+g) \cdot \left(-\frac{x+g}{y+f}\right)}{(y+f)^2}$$



$$y_2 = -\frac{\frac{(y+f)^2 + (x+g)^2}{y+f}}{(y+f)^2}$$

$$y_2 = -\frac{y^2 + f^2 + 2fy + x^2 + g^2 + 2gx}{(y+f)^3}$$

$$y_2 = -\frac{-c + f^2 + g^2}{(y+f)^3} \quad \left[\begin{array}{l} \because x^2 + y^2 + 2gx + 2fy + c = 0 \\ x^2 + y^2 + 2gx + 2fy = -c \end{array} \right]$$

$$y_2 = \frac{c - f^2 - g^2}{(y+f)^3}$$

Ans.

Q.4: Find y_4 if

(i) $y = \sin 3x$ (ii) $y = \cos^3 x$ (iii) $y = \ln(x^2 - 9)$

Solution:

(i) $y = \sin 3x$

$$y_1 = \cos 3x \cdot 3$$

$$y_1 = 3 \cos 3x$$

$$y_2 = 3(-\sin 3x) \cdot 3$$

$$y_2 = -9 \sin 3x$$

$$y_3 = -9 \cos 3x \cdot 3$$

$$y_3 = -27 \cos 3x$$

$$y_4 = -27(-\sin 3x) (3)$$

$$y_4 = 81 \sin 3x$$

Ans.

(ii) $y = \cos^3 x$ (1) (L.B 2008)

$$\cos 3x = \cos(2x + x)$$

$$\cos 3x = \cos 2x \cos x - \sin 2x \sin x$$

$$\cos 3x = (2\cos^2 x - 1) \cos x - 2\sin x \cos x \cdot \sin x$$

$$\cos 3x = 2\cos^3 x - \cos x - 2\cos x \sin^2 x$$

$$\cos 3x = 2\cos^3 x - \cos x - 2\cos x (1 - \cos^2 x)$$

$$\cos 3x = 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x$$

$$\cos 3x = 4\cos^3 x - 3\cos x$$

$$4\cos^3 x = \cos 3x + 3\cos x$$

$$\cos^3 x = \frac{1}{4}(\cos 3x + 3\cos x)$$

$\therefore$ From equation (1)



$$y = \frac{1}{4}(\cos 3x + 3 \cos x)$$

$$y_1 = \frac{1}{4}[-\sin 3x \cdot 3 + 3(-\sin x)]$$

$$y_1 = \frac{1}{4}[-3 \sin 3x - 3 \sin x]$$

$$y_2 = \frac{1}{4}[-3 \cos 3x \cdot 3 - 3 \cos x]$$

$$y_2 = \frac{1}{4}[-9 \cos 3x - 3 \cos x]$$

$$y_3 = \frac{1}{4}[-9(-\sin 3x) \cdot 3 - 3(-\sin x)]$$

$$y_3 = \frac{1}{4}[27 \sin 3x + 3 \sin x]$$

$$y_4 = \frac{1}{4}[27 \cos 3x \cdot 3 + 3 \cos x]$$

$$y_4 = \frac{1}{4}[81(4 \cos^3 x - 3 \cos x) + 3 \cos x]$$

$$y_4 = \frac{1}{4}[324 \cos^3 x - 243 \cos x + 3 \cos x]$$

$$y_4 = \frac{1}{4}[324 \cos^3 x - 240 \cos x]$$

$$y_4 = \frac{4}{4}[81 \cos^3 x - 60 \cos x]$$

$$\boxed{y_4 = -60 \cos x + 81 \cos^3 x}$$

Ans.

(iii) $y = \ln(x^2 - 9)$

$$y = \ln(x+3)(x-3)$$

$$y = \ln(x+3) + \ln(x-3)$$

$$y_1 = \frac{1}{(x+3)} + \frac{1}{(x-3)}$$

$$y_1 = (x+3)^{-1} + (x-3)^{-1}$$

$$y_2 = -(x+3)^{-2} - (x-3)^{-2}$$

$$y_3 = 2(x+3)^{-3} + 2(x-3)^{-3}$$

$$y_4 = -6(x+3)^{-4} - 6(x-3)^{-4}$$



$$y_4 = \frac{-6}{(x+3)^4} - \frac{6}{(x-3)^4}$$

$$\boxed{y_4 = -6 \left[\frac{1}{(x+3)^4} + \frac{1}{(x-3)^4} \right]} \quad \text{Ans.}$$

Q.5: If $x = \sin \theta$, $y = \sin m\theta$, Show that $(1-x^2)y_2 - xy_1 + m^2y = 0$
(G.B 2006)

Solution:

$$x = \sin \theta, \quad y = \sin m\theta$$

$$\theta = \sin^{-1}x$$

$$y = \sin m\theta$$

$$y = \sin m(\sin^{-1}x)$$

$$\sin^{-1}y = m \sin^{-1}x$$

$$\frac{1}{\sqrt{1-y^2}} \cdot y_1 = \frac{m}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2} y_1 = m \sqrt{1-y^2}$$

Squaring on both sides

$$(1-x^2)y_1^2 = m^2(1-y^2)$$

$$(1-x^2) \cdot 2y_1 y_2 + y_1^2 \cdot (-2x) = m^2(-2yy_1)$$

$$2y_1 [(1-x^2) \cdot y_2 - xy_1] = -2m^2yy_1$$

$$(1-x^2) \cdot y_2 - xy_1 = \frac{-2m^2yy_1}{2y_1}$$

$$(1-x^2) \cdot y_2 - xy_1 = -m^2y$$

$$\boxed{(1-x^2)y_2 - xy_1 + m^2y = 0} \quad \text{Hence proved.}$$

Q.6: If $y = e^x \sin x$, show that $\frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + 2y = 0$ (L.B 2009)

Solution:

$$y = e^x \sin x$$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = e^x \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(e^x)$$

$$\frac{dy}{dx} = e^x \cos x + \sin x e^x$$

$$= e^x (\cos x + \sin x)$$



Diff. again w.r.t. 'x'

$$\frac{d^2y}{dx^2} = e^x \frac{d}{dx} (\cos x + \sin x) + (\cos x + \sin x) \frac{d}{dx} (e^x)$$

$$\frac{d^2y}{dx^2} = e^x (-\sin x + \cos x) + (\cos x + \sin x) e^x$$

$$\frac{d^2y}{dx^2} = e^x (-\sin x + \cos x + \cos x + \sin x)$$

$$\frac{d^2y}{dx^2} = 2e^x \cos x$$

Taking

$$\begin{aligned} \frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + 2y &= 2e^x \cos x - 2e^x (\cos x + \sin x) + 2e^x \sin x \\ &= 2e^x \cos x - 2e^x \cos x - 2e^x \sin x + 2e^x \sin x \end{aligned}$$

$$\boxed{\frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + 2y = 0}$$

Hence proved.

Q.7: If $y = e^{ax} \sin bx$, show that $\frac{d^2y}{dx^2} - 2a \frac{dy}{dx} + (a^2 + b^2)y = 0$

Solution:

$$y = e^{ax} \sin bx$$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = e^{ax} \frac{d}{dx} (\sin bx) + \sin bx \frac{d}{dx} (e^{ax})$$

$$\frac{dy}{dx} = e^{ax} \cos bx \cdot b + \sin bx \cdot e^{ax} \cdot a$$

$$\frac{dy}{dx} = e^{ax} (b \cos bx + a \sin bx)$$

Diff. again w.r.t. 'x'

$$\frac{d^2y}{dx^2} = e^{ax} (-b \sin bx \cdot b + a \cos bx \cdot b) + (b \cos bx + a \sin bx) e^{ax} \cdot a$$

$$\frac{d^2y}{dx^2} = e^{ax} (-b^2 \sin bx + ab \cos bx + ab \cos bx + a^2 \sin bx)$$

$$\frac{d^2y}{dx^2} = e^{ax} (-b^2 \sin bx + 2ab \cos bx + a^2 \sin bx)$$

Taking

$$\frac{d^2y}{dx^2} - 2a \frac{dy}{dx} + (a^2 + b^2)y = e^{ax} (-b^2 \sin bx + 2ab \cos bx + a^2 \sin bx) - 2ae^{ax}$$



$$(b \cos bx + a \sin bx) + (a^2 + b^2) e^{ax} \sin bx$$

$$\frac{d^2y}{dx^2} - 2a \frac{dy}{dx} + (a^2 + b^2)y = -e^{ax} \sin bx + 2e^{ax} ab \cos bx + e^{ax} a^2 \sin bx - 2abe^{ax} \cos bx - 2a^2 e^{ax} \sin bx + a^2 e^{ax} \sin bx + b^2 e^{ax} \sin bx$$

$$\boxed{\frac{d^2y}{dx^2} - 2a \frac{dy}{dx} + (a^2 + b^2)y = 0} \text{ Hence proved.}$$

Q.8: If $y = (\cos^{-1} x)^2$, prove that $(1 - x^2) y_2 - xy_1 - 2 = 0$ (G.B 2007)

Solution:

$$y = (\cos^{-1} x)^2$$

$$y_1 = 2(\cos^{-1} x) \cdot \frac{-1}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2} y_1 = -2(\cos^{-1} x)$$

Squaring on both sides

$$(1-x^2) y_1^2 = 4(\cos^{-1} x)^2 \quad \because y = (\cos^{-1} x)^2$$

$$(1-x^2) y_1^2 = 4y$$

$$(1-x^2) \cdot 2y_1 y_2 + y_1^2 \cdot (-2x) = 4y_1$$

$$2y_1 [(1-x^2) y_2 - xy_1] = 4y_1$$

$$(1-x^2) y_2 - xy_1 = \frac{4y_1}{2y_1}$$

$$(1-x^2) y_2 - xy_1 = 2$$

$$\boxed{(1-x^2) y_2 - xy_1 - 2 = 0} \text{ Hence proved.}$$

Q.9: If $y = a \cos(\ln x) + b \sin(\ln x)$ Prove that $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$

Solution:

$$y = a \cos(\ln x) + b \sin(\ln x)$$

$$\frac{dy}{dx} = -a \sin(\ln x) \cdot \frac{1}{x} + b \cos(\ln x) \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{1}{x} [-a \sin(\ln x) + b \cos(\ln x)]$$

$$\frac{d^2y}{dx^2} = \frac{1}{x} \left[-a \cos(\ln x) \cdot \frac{1}{x} - b \sin(\ln x) \cdot \frac{1}{x} \right]$$

$$+ [(-a \sin(\ln x) + b \cos(\ln x))] \left(\frac{-1}{x^2} \right)$$



$$\frac{d^2y}{dx^2} = \frac{1}{x^2} [-a \cos(\ln x) - b \sin(\ln x) - \frac{1}{x^2} [-a \sin(\ln x) + b \cos(\ln x)]]$$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2} [-a \cos(\ln x) - b \sin(\ln x) + a \sin(\ln x) - b \cos(\ln x)]$$

Taking

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = x^2 \cdot \frac{1}{x^2} [-a \cos(\ln x) - b \sin(\ln x) + a \sin(\ln x)$$

$$- b \cos(\ln x)] + x \cdot \frac{1}{x} [-a \sin(\ln x) + b \cos(\ln x)] + a \cos(\ln x) + b \sin(\ln x)$$

$$= -a \cos(\ln x) - b \sin(\ln x) + a \sin(\ln x) - b \cos(\ln x) - a \sin(\ln x) + b \cos(\ln x) + a \cos(\ln x) + b \sin(\ln x)$$

$$\boxed{x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0}$$

Hence proved.



EXERCISE 2.8

Q.1 Apply the Maclaurin series expansion to prove that:

(i) $\ell n (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$ (L.B 2005)

(ii) $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots$

(iii) $\sqrt{1+x} = 1 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^3}{16} + \dots$

(iv) $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$ (L.B 20011)

(v) $e^{2x} = 1 + 2x + \frac{4x^2}{2} + \frac{8x^3}{6} + \dots$

Solution:

(i) $\ell n (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

Let

$$f(x) = \ell n (1 + x)$$

$$f(0) = \ell n (1 + 0) = \ell n 1 = 0$$



$$\begin{array}{l|l}
 f^I(x) = \frac{1}{1+x} & f^I(0) = \frac{1}{1+0} = 1 \\
 f^I(x) = (1+x)^{-1} & \\
 f^{II}(x) = -(1+x)^{-2} & f^{II}(0) = -(1+0)^{-2} = -1 \\
 f^{III}(x) = 2(1+x)^{-3} & f^{III}(0) = 2(1+0)^{-3} = 2 \\
 f^{IV}(x) = -6(1+x)^{-4} & f^{IV}(0) = -6(1+0)^{-4} = -6
 \end{array}$$

The Maclaurin series expansion is

$$\begin{aligned}
 f(x) &= f(0) + x f^I(0) + \frac{x^2}{2!} f^{II}(0) + \frac{x^3}{3!} f^{III}(0) + \frac{x^4}{4!} f^{IV}(0) + \dots \\
 &= 0 + x(1) + \frac{x^2}{2}(-1) + \frac{x^3}{6}(2) + \frac{x^4}{24}(-6) + \dots
 \end{aligned}$$

$$\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots} \quad \text{Hence proved.}$$

$$\text{(ii) } \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots$$

$$\begin{array}{l|l}
 \text{Let } f(x) = \cos x & \\
 f(0) = \cos 0 = 1 & \\
 f^I(x) = -\sin x & f^I(0) = -\sin 0 = 0 \\
 f^{II}(x) = -\cos x & f^{II}(0) = -\cos 0 = -1 \\
 f^{III}(x) = \sin x & f^{III}(0) = \sin 0 = 0 \\
 f^{IV}(x) = \cos x & f^{IV}(0) = \cos 0 = 1 \\
 f^V(x) = -\sin x & f^V(0) = -\sin 0 = 0 \\
 f^{VI}(x) = -\cos x & f^{VI}(0) = -\cos 0 = -1
 \end{array}$$

The Maclaurin series expansion is

$$\begin{aligned}
 f(x) &= f(0) + x f^I(0) + \frac{x^2}{2!} f^{II}(0) + \frac{x^3}{3!} f^{III}(0) + \frac{x^4}{4!} f^{IV}(0) + \frac{x^5}{5!} f^V(0) + \frac{x^6}{6!} f^{VI}(0) + \dots \\
 &= 1 + x(0) + \frac{x^2}{2}(-1) + \frac{x^3}{6}(0) + \frac{x^4}{24}(1) + \frac{x^5}{120}(0) + \frac{x^6}{720}(-1) + \dots
 \end{aligned}$$

$$\boxed{\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots} \quad \text{Hence proved.}$$

$$\text{(iii) } \sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \dots$$

$$\text{Let } f(x) = \sqrt{1+x}$$



$$\begin{aligned}
 f(0) &= \sqrt{1+0} \\
 &= \sqrt{1} = 1 \\
 f'(x) &= \frac{1}{2} (x+1)^{-\frac{1}{2}} & f'(0) &= \frac{1}{2} (1+0)^{-\frac{1}{2}} = \frac{1}{2} \\
 f''(x) &= \frac{-1}{4} (1+x)^{-\frac{3}{2}} & f''(0) &= \frac{-1}{4} (1+0)^{-\frac{3}{2}} = \frac{-1}{4} \\
 f'''(x) &= \frac{3}{8} (1+x)^{-\frac{5}{2}} & f'''(0) &= \frac{3}{8} (1+0)^{-\frac{5}{2}} = \frac{3}{8}
 \end{aligned}$$

The Maclaurin series expansion is

$$\begin{aligned}
 f(x) &= f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots \\
 &= 1 + x \left(\frac{1}{2} \right) + \frac{x^2}{2} \left(\frac{-1}{4} \right) + \frac{x^3}{6} \left(\frac{3}{8} \right) + \dots
 \end{aligned}$$

$$\boxed{\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \dots}$$

Hence proved.

$$\text{(iv) } e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$$

$$\begin{aligned}
 \text{Let } f(x) &= e^x \\
 f(0) &= e^0 = 1 \\
 f'(x) &= e^x & f'(0) &= e^0 = 1 \\
 f''(x) &= e^x & f''(0) &= e^0 = 1 \\
 f'''(x) &= e^x & f'''(0) &= e^0 = 1
 \end{aligned}$$

The Maclaurin series expansion is

$$\begin{aligned}
 f(x) &= f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots \\
 &= 1 + x(1) + \frac{x^2}{2} (1) + \frac{x^3}{3} (1) + \dots
 \end{aligned}$$

$$\boxed{e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots}$$

Hence proved.

$$\text{(v) } e^{2x} = 1 + 2x + \frac{4x^2}{2} + \frac{8x^3}{3} + \dots$$

Let

$$\begin{aligned}
 f(x) &= e^{2x} \\
 f(0) &= e^{2(0)} = e^0 = 1
 \end{aligned}$$



$$\begin{array}{l|l}
 f^I(x) = 2e^{2x} & f^I(0) = 2e^{2(0)} = 2e^0 = 2 \\
 f^{II}(x) = 4e^{2x} & f^{II}(0) = 4e^{2(0)} = 4e^0 = 4 \\
 f^{III}(x) = 8e^{2x} & f^{III}(0) = 8e^{2(0)} = 8e^0 = 8
 \end{array}$$

The Maclaurin series expansion is

$$f(x) = f(0) + x f^I(0) + \frac{x^2}{2!} f^{II}(0) + \frac{x^3}{3!} f^{III}(0) + \dots$$

$$e^{2x} = 1 + x(2) + \frac{x^2}{2!}(4) + \frac{x^3}{3!}(8) + \dots$$

$$e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \dots \quad \text{Hence proved.}$$

Q.2: Show that

$$\cos(x+h) = \cos x - h \sin x - \frac{h^2}{2!} \cos x + \frac{h^3}{3!} \sin x + \dots \text{ and evaluate } \cos 61^\circ.$$

Solution:

Let $f(x+h) = \cos(x+h)$
then

$$\begin{array}{lcl}
 f(x) & = & \cos x \\
 f^I(x) & = & -\sin x \\
 f^{II}(x) & = & -\cos x \\
 f^{III}(x) & = & \sin x
 \end{array}$$

The Taylor series expansion is

$$f(x+h) = f(x) + hf^I(x) + \frac{h^2}{2!} f^{II}(x) + \frac{h^3}{3!} f^{III}(x) + \dots$$

$$\cos(x+h) = \cos x + h(-\sin x) + \frac{h^2}{2!}(-\cos x) + \frac{h^3}{3!} \sin x + \dots$$

$$\cos(x+h) = \cos x - h \sin x - \frac{h^2}{2!} \cos x + \frac{h^3}{3!} \sin x + \dots$$

Put $x = 60^\circ$, $h = 1^\circ = \frac{\pi}{180} \text{ rad} = 0.01745 \text{ rad}$

$$\begin{aligned}
 \cos(60^\circ + 1^\circ) &= \cos 60^\circ - (0.01745) \sin 60^\circ \\
 &\quad - \frac{(0.01745)^2}{2} \cos 60^\circ + \frac{(0.01745)^3}{6} \sin 60^\circ + \dots
 \end{aligned}$$



$$\cos 61^\circ \approx 0.5 - 0.0151 - 0.000076 + 0.00000076 + \dots$$

$$\cos 61^\circ \approx 0.4848$$

Ans

Q.3: Show that

$$2^{x+h} = 2^x \left\{ 1 + (\ln 2) h + \frac{(\ln 2)^2}{2} h^2 + \frac{(\ln 2)^3}{3} h^3 + \dots \right\}$$

Solution:

Let

$$f(x+h) = 2^{x+h}$$

then

$$f(x) = 2^x$$

$$f'(x) = (\ln 2) 2^x$$

$$f''(x) = (\ln 2)^2 2^x$$

$$f'''(x) = (\ln 2)^3 2^x$$

The Taylor series expansion is

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{3} f'''(x) + \dots$$

$$2^{x+h} = 2^x + h (\ln 2) 2^x + \frac{h^2}{2} (\ln 2)^2 \cdot 2^x + \frac{h^3}{3} (\ln 2)^3 \cdot 2^x + \dots$$

$$2^{x+h} = 2^x \left\{ 1 + (\ln 2)h + \frac{(\ln 2)^2}{2} h^2 + \frac{(\ln 2)^3}{3} h^3 + \dots \right\}$$

Hence proved.



EXERCISE 2.9

Q.1: Determine the intervals in which f is increasing or decreasing for the domain mentioned in each case.

(i) $f(x) = \sin x$; $x \in (-\pi, \pi)$

(ii) $f(x) = \cos x$; $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (L.B 2005)

(iii) $f(x) = 4 - x^2$; $x \in (-2, 2)$

(iv) $f(x) = x^2 + 3x + 2$; $x \in (-4, 1)$

Solution:

$$f(x) = \sin x \quad ; \quad x \in (-\pi, \pi)$$

$$f'(x) = \cos x$$

Put

$$f'(x) = 0$$

$$\Rightarrow \cos x = 0$$

$$\Rightarrow x = \frac{-\pi}{2}, \frac{\pi}{2}$$

So the sub-intervals are

$$\left(-\pi, \frac{-\pi}{2}\right), \left(\frac{-\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \pi\right)$$

For $\left(-\pi, \frac{-\pi}{2}\right)$

$$f'(x) = \cos x < 0 \quad \text{in} \left(-\pi, \frac{-\pi}{2}\right)$$

$$\text{So } f(x) \text{ is decreasing in } \left(-\pi, \frac{-\pi}{2}\right)$$

For $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$$f'(x) = \cos x > 0 \quad \text{in} \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

$$\text{So } f(x) \text{ is increasing in } \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

For $\left(\frac{\pi}{2}, \pi\right)$



$$f'(x) = \cos x < 0 \quad \text{in } \left(\frac{\pi}{2}, \pi\right)$$

So $f(x)$ is decreasing in $\left(\frac{\pi}{2}, \pi\right)$

$$(ii) \quad f(x) = \cos x \quad ; \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$f'(x) = -\sin x$$

Put

$$f'(x) = 0$$

$$\sin x = 0$$

$$x = 0$$

So the sub-intervals are $\left(-\frac{\pi}{2}, 0\right), \left(0, \frac{\pi}{2}\right)$

For $\left(-\frac{\pi}{2}, 0\right)$

$$f'(x) = -\sin x > 0 \quad \text{in } \left(-\frac{\pi}{2}, 0\right)$$

So f is increasing in $\left(-\frac{\pi}{2}, 0\right)$

For $\left(0, \frac{\pi}{2}\right)$

$$f'(x) = -\sin x < 0 \quad \text{in } \left(0, \frac{\pi}{2}\right)$$

So f is decreasing in $\left(0, \frac{\pi}{2}\right)$

$$(iii) \quad f(x) = 4 - x^2 \quad ; \quad x \in (-2, 2) \quad (L.B \ 2008)$$

$$f'(x) = -2x$$

Put

$$f'(x) = 0$$

$$-2x = 0$$

$$x = 0$$

So the sub-intervals are $(-2, 0)$ and $(0, 2)$

For $(-2, 0)$

$$f'(x) = -2x > 0 \quad \text{in } (-2, 0)$$

So $f(x)$ is increasing in $(-2, 0)$

For $(0, 2)$

$$f'(x) = -2x < 0 \quad \text{in } (0, 2)$$



So $f(x)$ is decreasing in $(0, 2)$

(iv) $f(x) = x^2 + 3x + 2$; $x \in (-4, 1)$ (**L.B 2007**) (**G.B 2008**)

$$f'(x) = 2x + 3$$

Put $f'(x) = 0$

$$2x + 3 = 0$$

$$2x = -3$$

$$x = \frac{-3}{2}$$

So the sub-intervals are $\left(-4, \frac{-3}{2}\right)$, $\left(\frac{-3}{2}, 1\right)$

For $\left(-4, \frac{-3}{2}\right)$

$$f'(x) = 2x + 3 < 0 \text{ in } \left(-4, \frac{-3}{2}\right)$$

So $f(x)$ is decreasing in $\left(-4, \frac{-3}{2}\right)$

For $\left(\frac{-3}{2}, 1\right)$

$$f'(x) = 2x + 3 > 0 \text{ in } \left(\frac{-3}{2}, 1\right)$$

So $f(x)$ is increasing in $\left(\frac{-3}{2}, 1\right)$

Q.2: Find the extreme values of the following functions defined as

(i) $f(x) = 1 - x^3$

(ii) $f(x) = x^2 - x - 2$

(iii) $f(x) = 5x^2 - 6x + 2$

(iv) $f(x) = 3x^2$

(v) $f(x) = 3x^2 - 4x + 5$

(vi) $f(x) = 2x^3 - 2x^2 - 36x + 3$

(vii) $f(x) = x^4 - 4x^2$

(viii) $f(x) = (x - 2)^2 (x - 1)$

(ix) $f(x) = 5 + 3x - x^3$ (**L.B 2011**)

Solution:

(i) $f(x) = 1 - x^3$

$$f'(x) = -3x^2$$

$$f''(x) = -6x$$

For stationary points

Put $f'(x) = 0$

$$-3x^2 = 0$$



$$\begin{aligned}x^2 &= 0 \\x &= 0\end{aligned}$$

The second derivative does not help in determining the extreme values.

Before $x = 0$, $f'(x) < 0$

After $x = 0$, $f'(x) < 0$

$\therefore x = 0$ has a point of inflection.

Put

$$\begin{aligned}x &= 0 \text{ in} \\f(x) &= 1 - x^3 \\f(0) &= 1 - (0)^3 = 1\end{aligned}$$

$\therefore$ Point of inflection is $(0, 1)$

(ii) $f(x) = x^2 - x - 2$
 $f'(x) = 2x - 1$
 $f''(x) = 2$

For stationery points

Put $f'(x) = 0$
 $2x - 1 = 0$
 $2x = 1$
 $x = \frac{1}{2}$

Put $x = \frac{1}{2}$ in $f''(x)$, we get

$$f''\left(\frac{1}{2}\right) = 2 > 0$$

$\therefore f$ has relative minima at $x = \frac{1}{2}$

Put $x = \frac{1}{2}$ in

$$\begin{aligned}f(x) &= x^2 - x - 2 \\f\left(\frac{1}{2}\right) &= \left(\frac{1}{2}\right)^2 - \frac{1}{2} - 2 \\&= \frac{1}{4} - \frac{1}{2} - 2 \\&= \frac{1 - 2 - 8}{4} = \frac{-9}{4}\end{aligned}$$

Ans.



(iii)

(L.B 2009 (s)) (L.B 2009)

$$f(x) = 5x^2 - 6x + 2$$

$$f'(x) = 10x - 6$$

$$f''(x) = 10$$

For stationary points

Put

$$f'(x) = 0$$

$$10x - 6 = 0$$

$$10x = 6$$

$$x = \frac{6}{10}$$

$$= \frac{3}{5}$$

Put

$$x = \frac{3}{5} \text{ in } f''(x), \text{ we get}$$

$$f''\left(\frac{3}{5}\right) = 10 > 0$$

$$\therefore f \text{ has relative minima at } x = \frac{3}{5}$$

$$\text{Put } x = \frac{3}{5} \text{ in}$$

$$f(x) = 5x^2 - 6x + 2$$

$$f\left(\frac{3}{5}\right) = 5\left(\frac{3}{5}\right)^2 - 6\left(\frac{3}{5}\right) + 2$$

$$= 5\left(\frac{9}{25}\right) - \frac{18}{5} + 2$$

$$= \frac{9}{5} - \frac{18}{5} + 2$$

$$= \frac{9 - 18 + 10}{5}$$

$$\boxed{f\left(\frac{3}{5}\right) = \frac{1}{5}}$$

Ans

$$(iv) \quad f(x) = 3x^2$$



$$f'(x) = 6x$$

$$f''(x) = 6$$

For stationary points

Put $f'(x) = 0$

$$6x = 0$$

$$x = 0$$

Put $x = 0$ in $f''(x)$, we get

$$f''(0) = 6 > 0$$

$\therefore$ f has relative minima at $x = 0$

Put $x = 0$ in

$$f(x) = 3x^2$$

$$f(0) = 3(0)^2$$

$$\boxed{f(0) = 0} \text{ Ans}$$

(v)

(G.B 2008)

$$f(x) = 3x^2 - 4x + 5$$

$$f'(x) = 6x - 4$$

$$f''(x) = 6$$

For stationary points

Put $f'(x) = 0$

$$6x - 4 = 0$$

$$6x = 4$$

$$x = \frac{4}{6}$$

$$= \frac{2}{3}$$

Put $x = \frac{2}{3}$ in $f''(x)$, we get

$$f''\left(\frac{2}{3}\right) = 6 > 0$$

$\therefore$ f has relative minima at $x = \frac{2}{3}$

Put $x = \frac{2}{3}$ in

$$f(x) = 3x^2 - 4x + 5$$



$$\begin{aligned}
 f\left(\frac{2}{3}\right) &= 3\left(\frac{2}{3}\right)^2 - 4\left(\frac{2}{3}\right) + 5 \\
 &= 3\left(\frac{4}{9}\right) - \frac{8}{3} + 5 \\
 &= \frac{4}{3} - \frac{8}{3} + 5 \\
 &= \frac{4 - 8 + 15}{3}
 \end{aligned}$$

$$f\left(\frac{2}{3}\right) = \frac{11}{3} \text{ Ans}$$

(vi)

(G.B 2005)

$$f(x) = 2x^3 - 2x^2 - 36x + 3$$

$$f'(x) = 6x^2 - 4x - 36$$

$$f''(x) = 12x - 4$$

For Stationary Points

Put

$$f'(x) = 0$$

$$6x^2 - 4x - 36 = 0$$

$$2(3x^2 - 2x - 18) = 0$$

$$3x^2 - 2x - 18 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here $a = 3, \quad b = -2, \quad c = -18$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-18)}}{2(3)}$$

$$x = \frac{2 \pm \sqrt{4 + 216}}{6}$$

$$x = \frac{2 \pm \sqrt{220}}{6}$$

$$x = \frac{2 \pm \sqrt{4 \times 55}}{6}$$

$$x = \frac{2 \pm 2\sqrt{55}}{6}$$

$$x = \frac{2(1 \pm \sqrt{55})}{6}$$



$$x = \frac{1 \pm \sqrt{55}}{3}$$

Put $x = \frac{1 + \sqrt{55}}{3}$ in $f''(x)$, we get

$$\begin{aligned} f''\left(\frac{1 + \sqrt{55}}{3}\right) &= 12\left(\frac{1 + \sqrt{55}}{3}\right) - 4 \\ &= 4(1 + \sqrt{55}) - 4 \\ &= 4 + 4\sqrt{55} - 4 \\ &= 4\sqrt{55} > 0 \end{aligned}$$

$\therefore$ f has relative minima at $x = \frac{1 + \sqrt{55}}{3}$

Put $x = \frac{1 + \sqrt{55}}{3}$ in

$$\begin{aligned} f(x) &= 2x^3 - 2x^2 - 36x + 3 \\ f\left(\frac{1 + \sqrt{55}}{3}\right) &= 2\left(\frac{1 + \sqrt{55}}{3}\right)^3 - 2\left(\frac{1 + \sqrt{55}}{3}\right)^2 - 36\left(\frac{1 + \sqrt{55}}{3}\right) + 3 \\ &= 2 \frac{[1 + 55\sqrt{55} + 3\sqrt{55} + 3(55)]}{27} - 2 \left(\frac{1 + 55 + 2\sqrt{55}}{9}\right) - 12(1 + \sqrt{55}) + 3 \\ &= \frac{2(166 + 58\sqrt{55})}{27} - 2\left(\frac{56 + 2\sqrt{55}}{9}\right) - 12 - 12\sqrt{55} + 3 \\ &= \frac{332 + 116\sqrt{55}}{27} - \frac{112 + 4\sqrt{55}}{9} - 12\sqrt{55} - 9 \\ &= \frac{332 + 116\sqrt{55} - 3(112 + 4\sqrt{55}) - 324\sqrt{55} - 243}{27} \\ &= \frac{89 - 208\sqrt{55} - 336 - 12\sqrt{55}}{27} \end{aligned}$$

$$f\left(\frac{1 + \sqrt{55}}{3}\right) = \frac{1}{27} (-247 - 220\sqrt{55})$$

Put $x = \frac{1 - \sqrt{55}}{3}$ in $f''(x)$, we get

$$\begin{aligned} f''\left(\frac{1 - \sqrt{55}}{3}\right) &= 12\left(\frac{1 - \sqrt{55}}{3}\right) - 4 \\ &= 4(1 - \sqrt{55}) - 4 = 4 - 4\sqrt{55} - 4 = -4\sqrt{55} < 0 \end{aligned}$$



∴ f has relative maxima at $x = \frac{1 - \sqrt{55}}{3}$

Put $x = \frac{1 - \sqrt{55}}{3}$ in

$$\begin{aligned}
 f(x) &= 2x^3 - 2x^2 - 36x + 3 \\
 &= 2\left(\frac{1 - \sqrt{55}}{3}\right)^3 - 2\left(\frac{1 - \sqrt{55}}{3}\right)^2 - 36\left(\frac{1 - \sqrt{55}}{3}\right) + 3 \\
 &= \frac{2[1 - 55\sqrt{55} - 3\sqrt{55} + 3(55)]}{27} - 2\left(\frac{1 + 55 - 2\sqrt{55}}{9}\right) - 12(1 - \sqrt{55}) + 3 \\
 &= \frac{2(166 - 58\sqrt{55})}{27} - 2\left(\frac{56 - 2\sqrt{55}}{9}\right) - 12 + 12\sqrt{55} + 3 \\
 &= \frac{332 - 116\sqrt{55}}{27} - \frac{112 - 4\sqrt{55}}{9} + 12\sqrt{55} - 9 \\
 &= \frac{332 - 116\sqrt{55} - 3(112 - 4\sqrt{55}) + 324\sqrt{55} - 243}{27} \\
 &= \frac{89 + 208\sqrt{55} - 336 + 12\sqrt{55}}{27}
 \end{aligned}$$

$$f\left(\frac{1 - \sqrt{55}}{3}\right) = \frac{1}{27} (-247 + 220\sqrt{55})$$

$$\begin{aligned}
 \text{(vii)} \quad f(x) &= x^4 - 4x^2 \\
 f'(x) &= 4x^3 - 8x \\
 f''(x) &= 12x^2 - 8
 \end{aligned}$$

For stationary points

Put

$$\begin{aligned}
 f'(x) &= 0 \\
 4x^3 - 8x &= 0 \\
 4x(x^2 - 2) &= 0 \\
 x(x^2 - 2) &= 0
 \end{aligned}$$

Either

$$\begin{aligned}
 x = 0 \quad \text{or} \quad x^2 - 2 &= 0 \\
 x^2 &= 2 \\
 x &= \pm\sqrt{2}
 \end{aligned}$$

Put $x = 0$ in $f''(x)$, we get

$$f''(0) = 12(0)^2 - 8 = -8 < 0$$

∴ f has relative maxima at $x = 0$



$$\begin{aligned}\text{Put } x &= 0 \text{ in} \\ f(x) &= x^4 - 4x^2 \\ f(0) &= (0)^4 - 4(0)^2 \\ \boxed{f(0) &= 0}\end{aligned}$$

Put

$$\begin{aligned}x &= \sqrt{2} \text{ in } f''(x), \text{ we get} \\ f''(\sqrt{2}) &= 12(\sqrt{2})^2 - 8 \\ &= 12(2) - 8 \\ &= 24 - 8 \\ &= 16 > 0\end{aligned}$$

$\therefore$ f has relative minima at $x = \sqrt{2}$

$$\begin{aligned}\text{Put } x &= \sqrt{2} \text{ in} \\ f(x) &= x^4 - 4x^2 \\ f(\sqrt{2}) &= (\sqrt{2})^4 - 4(\sqrt{2})^2 \\ &= 4 - 4(2) \\ &= 4 - 8\end{aligned}$$

$$\boxed{f(\sqrt{2}) = -4}$$

Put

$$\begin{aligned}x &= -\sqrt{2} \text{ in } f''(x), \text{ we get} \\ f''(-\sqrt{2}) &= 12(-\sqrt{2})^2 - 8 \\ &= 12(2) - 8 \\ &= 24 - 8 \\ &= 16 > 0\end{aligned}$$

$\therefore$ $f(x)$ has relative minima at $x = -\sqrt{2}$

$$\begin{aligned}\text{Put } x &= -\sqrt{2} \text{ in} \\ f(x) &= x^4 - 4x^2 \\ f(-\sqrt{2}) &= (-\sqrt{2})^4 - 4(-\sqrt{2})^2 \\ &= 4 - 4(2) \\ &= 4 - 8\end{aligned}$$

$$\boxed{f(-\sqrt{2}) = -4}$$



(viii)

$$f(x) = (x-2)^2(x-1)$$

$$f(x) = (x^2 + 4 - 4x)(x-1)$$

$$f(x) = x^3 + 4x - 4x^2 - x^2 - 4 + 4x$$

$$f(x) = x^3 - 5x^2 + 8x - 4$$

$$f'(x) = 3x^2 - 10x + 8$$

$$f''(x) = 6x - 10$$

For stationary points

Put

$$f'(x) = 0$$

$$3x^2 - 10x + 8 = 0$$

$$3x^2 - 6x - 4x + 8 = 0$$

$$3x(x-2) - 4(x-2) = 0$$

$$(x-2)(3x-4) = 0$$

Either

$$x-2 = 0 \quad \text{or} \quad 3x-4 = 0$$

$$x = 2 \quad \quad \quad 3x = 4$$

$$x = \frac{4}{3}$$

Put

 $x = 2$ in $f''(x)$, we get

$$f''(2) = 6(2) - 10$$

$$= 12 - 10$$

$$= 2 > 0$$

 $\therefore$ f has relative minima at $x = 2$

Put

 $x = 2$ in

$$f(x) = (x-2)^2(x-1)$$

$$f(2) = (2-2)^2(2-1)$$

$$f(2) = (0)^2(1)$$

$$\boxed{f(2) = 0}$$

Put

 $x = \frac{4}{3}$ in $f''(x)$, we get

$$f''\left(\frac{4}{3}\right) = 6\left(\frac{4}{3}\right) - 10$$

$$= 8 - 10 = -2 < 0$$



$\therefore$ f has relative maxima at $x = \frac{4}{3}$

Put $x = \frac{4}{3}$ in

$$\begin{aligned} f(x) &= (x-2)^2(x-1) \\ f\left(\frac{4}{3}\right) &= \left(\frac{4}{3}-2\right)^2\left(\frac{4}{3}-1\right) \\ &= \left(\frac{4-6}{3}\right)^2\left(\frac{4-3}{3}\right) \\ &= \left(\frac{-2}{3}\right)^2\left(\frac{1}{3}\right) \\ &= \left(\frac{4}{9}\right)\left(\frac{1}{3}\right) \end{aligned}$$

$$\boxed{f\left(\frac{4}{3}\right) = \frac{4}{27}}$$

(ix)

$$f(x) = 5 + 3x - x^3$$

$$f'(x) = 3 - 3x^2$$

$$f''(x) = -6x$$

For stationery points

Put $f'(x) = 0$

$$3 - 3x^2 = 0$$

$$-3x^2 = -3$$

$$x^2 = \frac{-3}{-3}$$

$$x^2 = 1$$

$$x = \pm 1$$

Put $x = 1$ in $f''(x)$, we get

$$f''(1) = -6(1) = -6 < 0$$

$\therefore$ f has relative maxima at $x = 1$

Put $x = 1$ in

$$f(x) = 5 + 3x - x^3$$

$$f(1) = 5 + 3(1) - (1)^3$$

$$= 5 + 3 - 1$$

$$f(1) = 7$$

Put $x = -1$ in $f''(x)$, we get



$$f''(-1) = -6(-1) = 6 > 0$$

∴ f has relative minima at $x = -1$

Put $x = -1$ in

$$f(x) = 5 + 3x - x^3$$

$$f(-1) = 5 + 3(-1) - (-1)^3$$

$$= 5 - 3 + 1$$

$$f(-1) = 3$$

Q.3: Find the maximum and minimum values of the function defined by the following equation occurring in the interval $[0, 2\pi]$.

$$f(x) = \sin x + \cos x$$

Solution:

$$f(x) = \sin x + \cos x$$

$$f'(x) = \cos x - \sin x$$

$$f''(x) = -\sin x - \cos x$$

For stationary points

Put $f'(x) = 0$

$$\cos x - \sin x = 0$$

$$\cos x = \sin x$$

$$\frac{\sin x}{\cos x} = 1$$

$$\tan x = 1$$

Since tangent is positive then in 1st and 3rd quadrant with reference angle $\frac{\pi}{4}$.

$$x = \frac{\pi}{4}, \quad x = \pi + \frac{\pi}{4}$$

$$x = \frac{5\pi}{4}$$

Put $x = \frac{\pi}{4}$ in $f'(x)$, we get

$$f''\left(\frac{\pi}{4}\right) = -\sin \frac{\pi}{4} - \cos \frac{\pi}{4}$$

$$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$$

$$= \frac{-1-1}{\sqrt{2}}$$



$$= \frac{-2}{\sqrt{2}} < 0$$

∴ f has relative maxima at $x = \frac{\pi}{4}$

Put $x = \frac{\pi}{4}$ in

$$f(x) = \sin x + \cos x$$

$$f\left(\frac{\pi}{4}\right) = \sin \frac{\pi}{4} + \cos \frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$= \frac{1+1}{\sqrt{2}}$$

$$= \frac{2}{\sqrt{2}}$$

$$= \frac{\sqrt{2} \times \sqrt{2}}{\sqrt{2}}$$

$$f\left(\frac{\pi}{4}\right) = \sqrt{2}$$

Put $x = \frac{5\pi}{4}$ in $f''(x)$, we get

$$f''\left(\frac{5\pi}{4}\right) = -\sin \frac{5\pi}{4} - \cos \frac{5\pi}{4}$$

$$= \left(\frac{-1}{\sqrt{2}}\right) - \left(\frac{-1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$= \frac{1+1}{\sqrt{2}}$$

$$= \frac{2}{\sqrt{2}} > 0$$

∴ f has relative minima at $x = \frac{5\pi}{4}$



Put $x = \frac{5\pi}{4}$ in

$$\begin{aligned} f(x) &= \sin x + \cos x \\ f\left(\frac{5\pi}{4}\right) &= \sin \frac{5\pi}{4} + \cos \frac{5\pi}{4} \\ &= \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \\ &= \frac{-1-1}{\sqrt{2}} \\ &= \frac{-2}{\sqrt{2}} \\ &= \frac{-\sqrt{2} \times \sqrt{2}}{\sqrt{2}} \\ f\left(\frac{5\pi}{4}\right) &= -\sqrt{2} \end{aligned}$$

Q.4: Show that $y = \frac{\ln x}{x}$ has maximum value at $x = e$ (L.B 2005)

Solution:

$$\begin{aligned} y &= \frac{\ln x}{x} \\ \frac{dy}{dx} &= \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2} \\ \frac{dy}{dx} &= \frac{1 - \ln x}{x^2} \\ \frac{d^2y}{dx^2} &= \frac{x^2 \cdot \frac{-1}{x} - (1 - \ln x) \cdot 2x}{(x^2)^2} \\ \frac{d^2y}{dx^2} &= \frac{-x - 2x + 2x \ln x}{x^4} \\ \frac{d^2y}{dx^2} &= \frac{-3x + 2x \ln x}{x^4} \end{aligned}$$

For stationary points



Put $\frac{dy}{dx} = 0$

$$\frac{1 - \ln x}{x^2} = 0$$

$$1 - \ln x = 0$$

$$1 = \ln x$$

$$\ln x = \ln e$$

$$x = e \quad \because \ln e = 1$$

Put $x = e$ in $\frac{d^2y}{dx^2}$, we get

$$\frac{d^2y}{dx^2} = \frac{-3e + 2e}{e^4}$$

$$= \frac{-e}{e^4}$$

$$= \frac{-1}{e^3} < 0$$

Shows $y = \frac{\ln x}{x}$ has maximum value at $x = e$.

Q.5: Show that $y = x^x$ has a minimum value at $x = \frac{1}{e}$ (L.B 2006)

Solution:

$$y = x^x$$

Taking ' $\ln$ ' on both sides

$$\ln y = \ln x^x$$

$$\ln y = x \ln x$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{1}{x} + \ln x \cdot 1$$

$$\frac{dy}{dx} = y [1 + \ln x]$$

$$\frac{dy}{dx} = x^x (1 + \ln x)$$

$$\frac{d^2y}{dx^2} = x^x \frac{d}{dx} (1 + \ln x) + (1 + \ln x) \frac{d}{dx} (x^x)$$



$$\frac{d^2y}{dx^2} = x^x \cdot \frac{1}{x} + (1 + \ln x) \cdot x^x (1 + \ln x)$$

$$\frac{d^2y}{dx^2} = x^x \left[\frac{1}{x} + (1 + \ln x)^2 \right]$$

For stationary points

Put $\frac{dy}{dx} = 0$

$$x^x (1 + \ln x) = 0$$

$$1 + \ln x = 0, \quad x^x \neq 0$$

$$\ln e + \ln x = 0$$

$$\ln x = -\ln e$$

$$\ln x = \ln e^{-1}$$

$$x = e^{-1}$$

$$x = \frac{1}{e}$$

Put $x = \frac{1}{e}$ in $\frac{d^2y}{dx^2}$, we get

$$\frac{d^2y}{dx^2} = \left(\frac{1}{e}\right)^{\frac{1}{e}} \left[\frac{1}{e} + \left(1 + \ln \frac{1}{e}\right)^2 \right]$$

$$\frac{d^2y}{dx^2} = \left(\frac{1}{e}\right)^{\frac{1}{e}} [e + (1 + \ln e^{-1})^2]$$

$$\frac{d^2y}{dx^2} = \left(\frac{1}{e}\right)^{\frac{1}{e}} [e + (1 - \ln e)^2]$$

$$\frac{d^2y}{dx^2} = \left(\frac{1}{e}\right)^{\frac{1}{e}} [e + (1 - 1)^2]$$

$$\frac{d^2y}{dx^2} = e \left(\frac{1}{e}\right)^{\frac{1}{e}} > 0$$

Shows $y = x^x$ has a minimum value at $x = \frac{1}{e}$



EXERCISE 2.10

Q.1: Find two positive integers whose sum is 30 and their product will be maximum.

Solution:

Let 1st positive integer = x

2nd positive integer = $30 - x$

According to the condition

$$\begin{aligned} \text{Product} &= P = x(30 - x) \\ P &= 30x - x^2 \end{aligned}$$

$$\frac{dP}{dx} = 30 - 2x$$

$$\frac{d^2P}{dx^2} = -2$$

For stationary points

Put

$$\frac{dP}{dx} = 0$$

$$30 - 2x = 0$$

$$30 = 2x$$

$$x = \frac{30}{2} = 15$$

Put $x = 15$ in $\frac{d^2P}{dx^2}$, we get

$$\frac{d^2P}{dx^2} = -2 < 0$$

∴ Product is maximum at $x = 15$

1st positive integer = $x = 15$

2nd positive integer = $30 - x$

$$= 30 - 15$$

$$= 15 \quad \text{Ans.}$$

Q.2: Divide 20 into two parts so that the sum of their squares will be minimum.

Solution:

Let

1st part = x



$$\begin{aligned}
 2^{\text{nd}} \text{ part} &= 20 - x \\
 \text{Sum of square} &= S = x^2 + (20 - x)^2 \\
 S &= x^2 + 400 + x^2 - 40x \\
 S &= 2x^2 - 40x + 400 \\
 \frac{dS}{dx} &= 4x - 40 \\
 \frac{d^2S}{dx^2} &= 4
 \end{aligned}$$

For stationary points

Put

$$\begin{aligned}
 \frac{dS}{dx} &= 0 \\
 4x - 40 &= 0 \\
 4x &= 40 \\
 x &= \frac{40}{4} = 10
 \end{aligned}$$

Put $x = 10$ in $\frac{d^2S}{dx^2}$, we get

$$\frac{d^2S}{dx^2} = 4 > 0$$

$\therefore$ Sum of squares is minimum at $x = 10$

$$1^{\text{st}} \text{ part} = x = 10$$

$$\begin{aligned}
 2^{\text{nd}} \text{ part} &= 20 - x \\
 &= 20 - 10 \\
 &= 10
 \end{aligned}$$

Ans.

Q.3: Find two positive integers whose sum is 12 and the product of one with the square of the other will be maximum. (L.B 2009)

Solution:

Let

$$1^{\text{st}} \text{ positive integer} = x$$

$$2^{\text{nd}} \text{ positive integer} = 12 - x$$

According to the condition

$$\text{Product} = P = x(12 - x)^2$$

$$P = x(144 + x^2 - 24x)$$

$$P = 144x + x^3 - 24x^2$$



$$\frac{dP}{dx} = 144x + 3x^2 - 48x$$

$$\frac{d^2P}{dx^2} = 6x - 48$$

For stationary points

Put $\frac{dP}{dx} = 0$

$$144 + 3x^2 - 48x = 0$$

$$3(x^2 - 16x + 48) = 0$$

$$x^2 - 16x + 48 = 0$$

$$x^2 - 12x - 4x + 48 = 0$$

$$x(x - 12) - 4(x - 12) = 0$$

$$(x - 12)(x - 4) = 0$$

Either

$$x - 12 = 0 \quad \text{or} \quad x - 4 = 0$$

$$x = 12 \quad \quad \quad x = 4$$

Put

$$x = 12 \quad \text{in} \quad \frac{d^2P}{dx^2}, \text{ we get}$$

$$\frac{d^2P}{dx^2} = 6(12) - 48$$

$$= 72 - 48$$

$$= 24 > 0$$

This is not possible because product will be maximum.

Put $x = 4$ in $\frac{d^2P}{dx^2}$, we get

$$\frac{d^2P}{dx^2} = 6(4) - 48$$

$$= 24 - 48$$

$$= -24 < 0$$

$\therefore$ Product is maximum at $x = 4$

$$1^{\text{st}} \text{ positive integer} = x = 4$$

$$2^{\text{nd}} \text{ positive integer} = 12 - x$$

$$= 12 - 4 = 8 \text{ Ans.}$$

Q.4: The perimeter of a triangle is 16 centimeters. If one side of length 6cm, what are length of the other sides for maximum area of the triangle?

Solution:



Length of 1st side of triangle = 6 cm

Let

Length of 2nd side of triangle = x cm

Length of 3rd side of triangle = $16 - (6 + x)$
 $= 16 - 6 - x$
 $= (10 - x) \text{ cm}$

$$S = \frac{6 + x + 10 - x}{2}$$

$$S = \frac{16}{2} = 8 \text{ cm}$$

$$\text{Area} = A = \sqrt{S(S-a)(S-b)(S-c)}$$

$$A = \sqrt{8(8-6)(8-x)(8-10+x)}$$

$$A = \sqrt{8(2)(8-x)(x-2)}$$

$$A = \sqrt{16(8x-16-x^2+2x)}$$

$$A = 4\sqrt{-x^2+10x-16}$$

$$\frac{dA}{dx} = 4 \cdot \frac{1}{2} \cdot (-x^2+10x-16)^{\frac{-1}{2}} (-2x+10)$$

$$\frac{dA}{dx} = \frac{2(-2x+10)}{\sqrt{-x^2+10x-16}}$$

For stationary points

Put

$$\frac{dA}{dx} = 0$$

$$\frac{2(-2x+10)}{\sqrt{-x^2+10x-16}} = 0$$

$$4(-x+5) = 0$$

$$-x+5 = 0$$

$$-x = -5$$

$$\boxed{x = 5}$$

Before $x = 5$, $\frac{dA}{dx} > 0$

After $x = 5$, $\frac{dA}{dx} < 0$

$\therefore$ Area is maximum at $x = 5$
 1st side of triangle = 6 cm



$$\begin{aligned} 2^{\text{nd}} \text{ side of triangle} &= x = 5 \text{ cm} \\ 3^{\text{rd}} \text{ side of triangle} &= 10 - x = 5 \text{ cm} \quad \text{Ans.} \end{aligned}$$

Q.5: Find the dimensions of a rectangle of largest area having perimeter 120 centimetres.

Solution:

Let

$$\text{Length of rectangle} = x \text{ cm}$$

$$\text{Width of rectangle} = y \text{ cm}$$

$$\text{Perimeter} = 120 \text{ cm}$$

$$2(x + y) = 120$$

$$x + y = \frac{120}{2}$$

$$x + y = 60$$

$$y = 60 - x \quad \dots (1)$$

$$\text{Area} = A = xy$$

$$A = x(60 - x)$$

$$A = 60x - x^2$$

$$\frac{dA}{dx} = 60 - 2x$$

$$\frac{d^2A}{dx^2} = -2$$

For Stationary points

Put

$$\frac{dA}{dx} = 0$$

$$60 - 2x = 0$$

$$-2x = -60$$

$$x = \frac{60}{2} = 30$$

Put $x = 30$ in $\frac{d^2A}{dx^2}$, we get

$$\frac{d^2A}{dx^2} = -2 < 0$$

$\therefore$ Area is maximum at $x = 30$

Put

$$x = 30 \text{ cm in eq. (1)}$$



$$y = 60 - 30 = 30 \text{ cm}$$

Length of rectangle = $x = 30 \text{ cm}$

Width of rectangle = $y = 30 \text{ cm}$

Dimensions of rectangle are 30 cm, 30 cm

Q.6: Find the lengths of the sides of a variable rectangle having area 36 cm^2 when its perimeter is minimum.

Solution:

Let

Length of rectangle = $x \text{ cm}$

Width of rectangle = $y \text{ cm}$

$$\text{Area} = A = 36 \text{ cm}^2$$

$$xy = 36$$

$$y = \frac{36}{x} \dots\dots\dots (1)$$

$$\text{Perimeter} = P = 2(x + y)$$

$$P = 2\left(x + \frac{36}{x}\right)$$

$$\frac{dP}{dx} = 2\left(1 - \frac{36}{x^2}\right)$$

$$\frac{d^2P}{dx^2} = 2\left(0 + \frac{72}{x^3}\right)$$

$$= \frac{144}{x^3}$$

For stationary points

$$\text{Put } \frac{dP}{dx} = 0$$

$$2\left(1 - \frac{36}{x^2}\right) = 0$$

$$1 - \frac{36}{x^2} = 0$$

$$\frac{x^2 - 36}{x^2} = 0$$

$$x^2 - 36 = 0$$

$$x^2 = 36$$

$$x = 6$$

$$\text{Put } x = 6 \text{ in } \frac{d^2P}{dx^2}, \text{ we get}$$



$$\frac{d^2P}{dx^2} = \frac{144}{(6)^3} = \frac{144}{216} > 0$$

∴ Perimeter is minimum at $x = 6$ cm

Put $x = 6$ cm in eq. (1)

$$y = \frac{36}{6} = 6 \text{ cm}$$

Length of rectangle = $x = 6$ cm

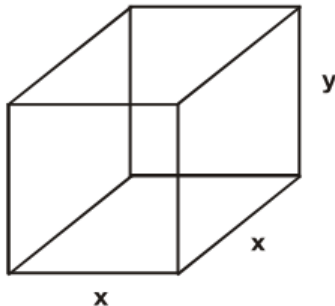
Width of rectangle = $y = 6$ cm Ans.

Ustad360.com



Q.7: A box with a square base and open top is to have a volume of 4 cubic dm.
Find the dimensions of the box which will require the least material.

Solution:



Let Length of box = x
 Width of box = x
 Height of box = y
 Volume = 4 cubic dm
 $x \times x \times y = 4$
 $x^2 y = 4$
 $y = \frac{4}{x^2}$ (1)

Surface area of box = $S = x^2 + xy + xy + xy + xy$
 $= x^2 + 4xy$
 $= x^2 + 4x \left(\frac{4}{x^2} \right)$

$$S = x^2 + \frac{16}{x}$$

$$\frac{dS}{dx} = 2x - \frac{16}{x^2}$$

$$\frac{d^2S}{dx^2} = 2 + \frac{32}{x^3}$$

For stationery points

Put $\frac{dS}{dx} = 0$

$$2x - \frac{16}{x^2} = 0$$

$$\frac{2x^3 - 16}{x^2} = 0$$

$$2x^3 - 16 = 0$$

$$2x^3 = 16$$



$$x^3 = 8$$

$$x = 2$$

Put $x = 2$ in $\frac{d^2S}{dx^2}$, we get

$$\frac{d^2S}{dx^2} = 2 + \frac{32}{(2)^3}$$

$$= 2 + \frac{32}{8}$$

$$= 2 + 4$$

$$= 12 > 0$$

∴ Surface area is minimum at $x = 2$

Put $x = 2$ in

$$y = \frac{4}{x^2} = \frac{4}{(2)^2} = \frac{4}{4} = 1$$

Length of box $= x = 2\text{dm}$

Width of box $= x = 2\text{dm}$

Height of box $= y = 1\text{dm}$ Ans.

Q.8: Find the dimensions of a rectangular garden having perimeter 80 metres if its area is to be maximum.

Solution:

Let Length of rectangular garden $= x \text{ m}$

Width of rectangular garden $= y \text{ m}$

Perimeter $= 80 \text{ m}$

$$2(x + y) = \frac{80}{2}$$

$$y = 40 - x \quad \dots (1)$$

$$\text{Area} = A = xy$$

$$A = x(40 - x)$$

$$A = 40x - x^2$$

$$\frac{dA}{dx} = 40 - 2x$$

$$\frac{d^2A}{dx^2} = -2$$

For stationary points

$$\text{Put } \frac{dA}{dx} = 0$$



$$\begin{aligned}
 40 - 2x &= 0 \\
 -2x &= -40 \\
 x &= \frac{-40}{-2} = 20
 \end{aligned}$$

Put $x = 20$ in $\frac{d^2A}{dx^2}$, we get

$$\frac{d^2A}{dx^2} = -2 < 0$$

∴ Area is maximum at $x = 20$

Put $x = 20$ in eq. (1)

$$y = 40 - x$$

$$y = 40 - 20 = 20$$

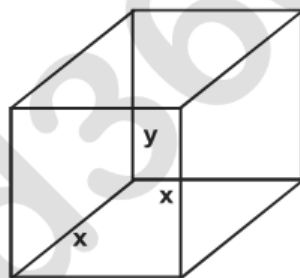
∴ Length of rectangular garden = $x = 20$ m

Width of rectangular garden = $y = 20$ m

So the dimensions of the rectangular garden are 20 m, 20 m.

Q.9: An open tank of square base of side x and vertical sides is to be constructed to contain of given quantity of water. Find the depth in terms of x if the expense of lining the inside of the tank with will be least.

Solution:



Let

Length of tank = x

Width of tank = x

Height of tank = y

Let 'q' be the quantity of water in the tank.

Volume = $V = q$

$x \times x \times y = q$

$$y = \frac{q}{x^2} \dots\dots\dots (1)$$

Total surface area = $S = x^2 + xy + xy + xy + xy$

$$S = x^2 + 4xy$$



$$S = x^2 + 4x \left(\frac{q}{x^2} \right)$$

$$S = x^2 + \frac{4q}{x}$$

$$\frac{dS}{dx} = 2x - \frac{4q}{x^2}$$

$$\frac{d^2S}{dx^2} = 2x + \frac{8q}{x^3}$$

For stationary points

Put $\frac{dS}{dx} = 0$

$$2x - \frac{4q}{x^2} = 0$$

$$\frac{2x^3 - 4q}{x^2} = 0$$

$$2x^3 - 4q = 0$$

$$2x^3 = 4q$$

$$x^3 = \frac{4q}{2}$$

$$x^3 = 2q$$

$$x = (2q)^{\frac{1}{3}}$$

Put $x = (2q)^{\frac{1}{3}}$ in

$$\frac{d^2S}{dx^2} = 2 + \frac{8q}{[(2q)^{\frac{1}{3}}]^3}$$

$$= 2 + \frac{8q}{2q}$$

$$= 2 + 4$$

$$= 6 > 0$$

$\therefore$ Surface area is minimum at $x = (2q)^{\frac{1}{3}}$



Now

$$x^3 = 2q$$

$$q = \frac{x^3}{2}$$

Put $q = \frac{x^3}{2}$ in eq. (1)

$$y = \frac{\frac{x^3}{2}}{x^2}$$

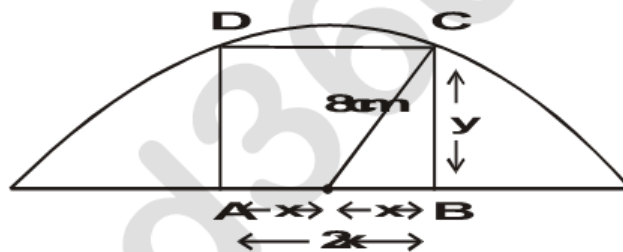
Depth	=	y	=	$\frac{x}{2}$
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Ans

Q.10: Find the dimensions of the rectangle of maximum area which fits, inside the semi-circle of radius 8 cm as shown in the figure.



Solution:



Let

Length of rectangle = $2x$ cm

Width of rectangle = y cm

From right angle $\triangle EBC$

$$(8)^2 = x^2 + y^2$$

$$y^2 = 64 - x^2$$

$$y = \sqrt{64 - x^2} \quad \dots (1)$$

Area = $A = (2x)(y)$

$$A = 2x \sqrt{64 - x^2}$$



Diff. w.r.t. 'x'

$$\frac{dA}{dx} = 2 \left[x \cdot \frac{1}{2} (64 - x^2)^{-\frac{1}{2}} - 2x + \sqrt{64 - x^2} \right]$$

$$\frac{dA}{dx} = 2 \left[\frac{-x^2}{\sqrt{64 - x^2}} + \sqrt{64 - x^2} \right]$$

$$\frac{dA}{dx} = 2 \left[\frac{-x^2 + 64 - x^2}{\sqrt{64 - x^2}} \right]$$

$$\frac{dA}{dx} = 2 \left(\frac{64 - 2x^2}{\sqrt{64 - x^2}} \right)$$

For stationary points

Put $\frac{dA}{dx} = 0$

$$2 \left(\frac{64 - 2x^2}{\sqrt{64 - x^2}} \right) = 0$$

$$64 - 2x^2 = 0$$

$$64 = 2x^2$$

$$x^2 = \frac{64}{2}$$

$$x^2 = 32$$

$$x = \sqrt{32}$$

$$x = 4\sqrt{2}$$

Before $x = 4\sqrt{2}$, $\frac{dA}{dx} > 0$

After $x = 4\sqrt{2}$, $\frac{dA}{dx} < 0$

$\therefore$ Area is maximum at $x = 4\sqrt{2}$

Put $x = 4\sqrt{2}$ in eq. (1)

$$y = \sqrt{64 - (4\sqrt{2})^2}$$

$$= \sqrt{64 - 32}$$

$$= \sqrt{32} = 4\sqrt{2}$$



$$\text{Length of rectangle} = x = 4\sqrt{2} \text{ cm}$$

$$\text{Width of rectangle} = y = 4\sqrt{2} \text{ cm}$$

$$\text{So the dimensions of rectangle are } 4\sqrt{2} \text{ cm, } 4\sqrt{2} \text{ cm}$$

Q.11: Find the point on the curve $y = x^2 - 1$ that is closest to the point $(3, -1)$.

Solution:

$$y = x^2 - 1$$

Let P (x, y) be the point on the curve and A (3, -1) be the given point.

Now Distance = $l = |PA|$

$$l = \sqrt{(x-3)^2 + (y+1)^2}$$

$$l = \sqrt{x^2 + 9 - 6x + (x^2 - 1 + 1)^2}$$

$$l = \sqrt{x^2 + 9 - 6x + x^4}$$

$$l = \sqrt{x^4 + x^2 - 6x + 9}$$

$$\frac{d\ell}{dx} = \frac{1}{2} (x^4 + x^2 - 6x + 9)^{-\frac{1}{2}} (4x^3 + 2x - 6)$$

$$= \frac{2(2x^3 + x - 3)}{2\sqrt{x^4 + x^2 - 6x + 9}}$$

$$\frac{d\ell}{dx} = \frac{2x^3 + x - 3}{\sqrt{x^4 + x^2 - 6x + 9}}$$

For stationary points

Put $\frac{d\ell}{dx} = 0$

$$\frac{2x^3 + x - 3}{\sqrt{x^4 + x^2 - 6x + 9}} = 0$$

$$2x^3 + x - 3 = 0$$

By using synthetic division

	2	0	1	-3
1		2	2	3
	2	2	3	0



$x = 1$ and depressed equation is $2x^2 + 2x + 3 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = 2, \quad b = 2, \quad c = 3$$

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(2)(3)}}{2(2)}$$

$$x = \frac{-2 \pm \sqrt{4 - 24}}{4}$$

$$x = \frac{-2 \pm \sqrt{-20}}{4}$$

Neglecting because it has imaginary roots.

Before $x = 1$, $\frac{d\ell}{dx} < 0$

After $x = 1$, $\frac{d\ell}{dx} > 0$

$\therefore$ Distance is minimum at $x = 1$

Put $x = 1$ in

$$y = x^2 - 1$$

$$y = (1)^2 - 1 = 1 - 1 = 0$$

$\therefore$ Point is $P(x, y) = P(1, 0)$ Ans.

Q.12: Find the point on the curve $y = x^2 + 1$ that is closest to the point (18, 1).

Solution:

$$y = x^2 + 1$$

Let $P(x, y)$ be the point on the curve and $A(18, 1)$ be the given point.

Now

$$\text{Distance} = \ell = |PA|$$

$$\ell = \sqrt{(x - 18)^2 + (y - 1)^2}$$

$$\ell = \sqrt{x^2 + 324 - 36x + (x^2 + 1 - 1)^2}$$

$$\ell = \sqrt{x^2 + 324 - 36x + x^4}$$



$$l = \sqrt{x^4 + x^2 - 36x + 324}$$

$$\frac{dl}{dx} = \frac{1}{2} (x^4 + x^2 - 36x + 324)^{-\frac{1}{2}} \cdot (4x^3 + 2x - 36)$$

$$\frac{dl}{dx} = \frac{2(2x^3 + x - 18)}{2\sqrt{x^4 + x^2 - 36x + 324}}$$

$$\frac{dl}{dx} = \frac{2x^3 + x - 18}{\sqrt{x^4 + x^2 - 36x + 324}}$$

For stationary points

Put

$$\frac{dl}{dx} = 0$$

$$\frac{2x^3 + x - 18}{\sqrt{x^4 + x^2 - 36x + 324}} = 0$$

$$2x^3 + x - 18 = 0$$

By using synthetic division

	2	0	1	-18
2		4	8	8
	2	4	9	0

$x = 2$ and depressed equation is $2x^2 + 4x + 9 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = 2, \quad b = 4, \quad c = 9$$

$$x = \frac{-4 \pm \sqrt{(4)^2 - 4(2)(9)}}{2(2)}$$

$$x = \frac{-4 \pm \sqrt{16 - 72}}{4}$$



$$x = \frac{-4 \pm \sqrt{-56}}{4}$$

Neglecting because it has imaginary roots.

Before $x = 2$, $\frac{d\ell}{dx} < 0$

After $x = 2$, $\frac{d\ell}{dx} > 0$

$\therefore$ Distance is minimum at $x = 2$

Put $x = 2$ in

$$y = x^2 + 1$$

$$y = (2)^2 + 1 = 4 + 1 = 5$$

$\therefore$ Point is $P(x, y) = P(2, 5)$ Ans.

