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CHAPTER

3

INTEGRATION

EXERCISE 3.1

Q.1 Find δy and dy in the following cases.

(i) $y = x^2 - 1$ when x changes from 3 to 3.02 (Lhr.Board 2008, 2011, Guj.Board 2008)

(ii) $y = x^2 + 2x$ when x changes from 2 to 1.8

(iii) $y = \sqrt{x}$ when x changes from 4 to 4.41 (Lhr. Board 2005)

Solution:

(i) $y = x^2 - 1$ when x changes from 3 to 3.02

$$x = 3, \quad \delta x = 3.02 - 3 = 0.02$$

$$\begin{aligned}
 y + \delta y &= (x + \delta x)^2 - 1 - y \\
 &= (x + \delta x)^2 - 1 - (x^2 - 1) \\
 &= (x + \delta x)^2 - 1 - x^2 + 1 \\
 &= (x + \delta x)^2 - x^2 \\
 &= (3 + 0.02)^2 - (3)^2 \\
 &= (3.02)^2 - 9 \\
 &= 9.1204 - 9 \\
 &= 0.1204 \quad \text{Ans.}
 \end{aligned}$$

$$y = x^2 - 1$$

Taking differential on both sides

$$dy = d(x^2) - d(1)$$

$$dy = 2x dx \quad (\because dx = \delta x)$$

$$= 2(3)(0.02)$$



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$$dy = 0.12 \quad \text{Ans.}$$

(ii) $y = x^2 + 2x$ when x changes from 2 to 1.8

$$x = 2, \quad \delta x = 1.8 - 2 = -0.2$$

$$y + \delta y = (x + \delta x)^2 + 2(x + \delta x)$$

$$\delta y = (x + \delta x)^2 + 2x + 2\delta x - y$$

$$\delta y = (x + \delta x)^2 + 2x + 2\delta x - (x^2 + 2x)$$

$$= (x + \delta x)^2 + 2x + 2\delta x - x^2 - 2x$$

$$= (x + \delta x)^2 + 2\delta x - x^2$$

$$= (2 - 0.2)^2 + 2(-0.2) - (2)^2$$

$$= (1.8)^2 - 0.4 - 4$$

$$= 3.24 - 4.4$$

$$\delta y = -1.16 \quad \text{Ans.}$$

$$y = x^2 + 2x$$

Taking differential on both sides

$$dy = d(x^2) + 2dx$$

$$dy = 2xdx + 2dx$$

$$= 2(2)(-0.2) + 2(-0.2) \quad (\because \delta x = dx)$$

$$= -0.8 - 0.4$$

$$dy = -1.2 \quad \text{Ans}$$

(iii) $y = \sqrt{x}$ when x changes from 4 to 4.41

$$x = 4, \quad \delta x = 4.41 - 4 = 0.41$$

$$y + \delta y = \sqrt{x + \delta x}$$

$$\delta y = \sqrt{x + \delta x} - y$$

$$\delta y = \sqrt{x + \delta x} - \sqrt{x}$$

$$= \sqrt{4 + 0.41} - \sqrt{4}$$

$$= \sqrt{4.41} - 2$$

$$= 2.1 - 2 = 0.1 \quad \text{Ans.}$$

$$y = \sqrt{x}$$



Taking differential on both sides

$$dy = d(\sqrt{x})$$

$$dy = \frac{1}{2} x^{-1/2} dx$$

$$dy = \frac{1}{2\sqrt{x}} dx$$

$$dy = \frac{0.41}{2\sqrt{4}} \quad (\because \delta x = dx)$$

$$dy = \frac{0.41}{4} = 0.1025 \quad \text{Ans.}$$

Q.2 Using differentials find $\frac{dy}{dx}$ and $\frac{dx}{dy}$ in the following equations.

(i) $xy + x = 4$

(ii) $x^2 + 2y^2 = 16$

(iii) $x^4 + y^2 = xy^2$

(iv) $xy - \ln x = c$

Solution:

(i) $xy + x = 4$ (Guj. Board 2008)

Taking differential on both sides

$$d(xy) + dx = d(4)$$

$$xdy + ydx + dx = 0$$

$$xdy = -ydx - dx$$

$$dy = -\frac{(y+1)dx}{x}$$

$$\frac{dy}{dx} = -\frac{y+1}{x} \quad \text{Ans.}$$

$$\frac{dx}{dy} = -\frac{x}{y+1} \quad \text{Ans.}$$

(ii) $x^2 + 2y^2 = 16$ (Lhr. Board 2008)

Taking differential on both sides

$$d(x^2) + 2d(y^2) = d(16)$$

$$2xdx + 4ydy = 0$$



$$4ydy = -2xdx$$

$$\frac{dy}{dx} = \frac{-2x}{4y}$$

$$\frac{dy}{dx} = -\frac{x}{2y} \text{ Ans.}$$

$$\frac{dx}{dy} = -\frac{2y}{x} \text{ Ans.}$$

(iii) $x^4 + y^2 = xy^2$

Taking differential on both sides

$$d(x^4) + d(y^2) = d(xy^2)$$

$$4x^3dx + 2ydy = x d(y^2) + y^2dx$$

$$4x^3dx + 2ydy = 2xy dy + y^2dx$$

$$2ydy - 2xydy = y^2dx - 4x^3dx$$

$$2y(1-x)dy = (y^2 - 4x^3)dx$$

$$\frac{dy}{dx} = \frac{y^2 - 4x^3}{2y(1-x)} \text{ Ans.}$$

$$\frac{dx}{dy} = \frac{2y(1-x)}{y^2 - 4x^3}$$

$$\frac{dx}{dy} = \frac{2y(x-1)}{4x^3 - y^2} \text{ Ans.}$$

(iv) $xy - \ln x = c$

Taking differential on both sides

$$d(xy) - d(\ln x) = d(c)$$

$$xdy + ydx - \frac{1}{x}dx = 0$$

$$xdy = \frac{1}{x}dx - ydx$$

$$xdy = \left(\frac{1}{x} - y\right)dx$$

$$xdy = \left(\frac{1-xy}{x}\right)dx$$



$$\frac{dy}{dx} = \frac{1 - xy}{x^2}$$

$$\frac{dx}{dy} = \frac{x^2}{1 - xy} \quad \text{Ans.}$$

Q.3 Use differentials to approximate the values of

(i) $\sqrt[4]{17}$ (ii) $(31)^{1/5}$ (iii) $\cos 29^\circ$ (iv) $\sin 61^\circ$

Solution:

(i) $\sqrt[4]{17}$

Let

$$y = f(x) = \sqrt[4]{x} \text{ with } x = 16, \delta x = dx = 1$$

$$dy = f'(x) dx$$

$$= \frac{1}{4} x^{-3/4} dx$$

$$= \frac{dx}{4x^{3/4}} = \frac{1}{4(16)^{3/4}}$$

$$= \frac{1}{4(2^4)^{3/4}}$$

$$= \frac{1}{4(8)} = \frac{1}{32}$$

$$dy = 0.03125$$

$$f(x) = \sqrt[4]{x}$$

$$f(16) = \sqrt[4]{16}$$

$$= (2^4)^{1/4}$$

$$= 2$$

Using

$$f(x + \delta x) \approx f(x) + dy$$

$$f(16 + 1) \approx f(16) + dy$$

$$f(17) \approx 2 + 0.03125$$



$$\sqrt[4]{17} \approx 2.03125 \quad \text{Ans.}$$

(ii) $(31)^{1/5}$

Let

$$y = f(x) = x^{1/5} \quad \text{with } x = 32, \delta x = dx = -1$$

$$dy = f'(x) dx$$

$$= \frac{1}{5} x^{-4/5} dx$$

$$= \frac{dx}{5x^{4/5}}$$

$$= \frac{-1}{5(32)^{4/5}}$$

$$= \frac{-1}{5(2^5)^{4/5}}$$

$$= \frac{-1}{5(16)}$$

$$= \frac{-1}{80}$$

$$dy = -0.0125$$

$$f(x) = x^{1/5}$$

$$f(32) = (32)^{1/5}$$

$$= (2^5)^{1/5}$$

$$= 2$$

Using $f(x + \delta x) \approx f(x) + dy$

$$f(32 - 1) \approx f(32) + dy$$

$$f(31) \approx 2 - 0.0125$$

$$(31)^{1/5} \approx 1.9875 \quad \text{Ans.}$$

(iii) $\cos 29^\circ$

Let

$$y = f(x) = \cos x \quad \text{with } x = 30^\circ, \delta x = dx = -1^\circ$$



$$= -1 \times \frac{\pi}{180} = -0.0174$$

$$\begin{aligned} dy &= f'(x) dx \\ &= -\sin x dx \\ &= -\sin 30^\circ \times (-0.0174) \end{aligned}$$

$$dy = 0.0087$$

$$f(x) = \cos x$$

$$\begin{aligned} f(30^\circ) &= \cos 30^\circ \\ &= 0.8660 \end{aligned}$$

Using

$$\begin{aligned} f(x + \delta x) &\approx f(x) + dy \\ f(30^\circ - 1^\circ) &\approx f(30^\circ) + dy \\ f(29^\circ) &\approx 0.8660 + 0.0087 \\ \cos 29^\circ &\approx 0.8747 \quad \text{Ans.} \end{aligned}$$

(iv) **sin 61°**

Let

$$\begin{aligned} y &= f(x) = \sin x \text{ with } x = 60^\circ, dx = \delta x = 1^\circ \\ &= 1 \times \frac{\pi}{180} = 0.0174 \end{aligned}$$

$$\begin{aligned} dy &= f'(x) dx \\ &= \cos x dx \\ &= \cos 60^\circ \times 0.0174 \end{aligned}$$

$$dy = 0.0087$$

$$f(x) = \sin x$$

$$\begin{aligned} f(60^\circ) &= \sin 60^\circ \\ &= 0.8660 \end{aligned}$$

Using

$$\begin{aligned} f(x + \delta x) &\approx f(x) + dy \\ f(60^\circ + 1^\circ) &\approx f(60^\circ) + dy \end{aligned}$$



$$f(61^\circ) \approx 0.8660 + 0.0087$$

$$\sin 61^\circ \approx 0.8747 \quad \text{Ans.}$$

Q.4 Find the approximate increase in the volume of a cube if the length of its each edge changes from 5 to 5.02

Solution:

Let x be the each edge of cube. Since the length of each edge changes from 5 to 5.02

$$\therefore \quad x = 5, \quad dx = \delta x = 5.02 - 5 \\ = 0.02$$

$$\text{Volume of cube} = V = x \times x \times x \\ V = x^3$$

$$dv = 3x^2 dx$$

$$dv = 3(5)^2 (0.02)$$

$$dv = 1.5 \text{ cubic unit}$$

$$\text{Increase in volume} = dv = 1.5 \text{ cubic unit}$$

Q.5 Find the approximate increase in the area of a circular disc if its diameter is increased from 44cm to 44.4 cm

Solution:

Let r be the radius of circular disc. Since diameter increased from 44 cm to 44.4 cm so radius is

$$\frac{44}{2} \text{ cm to } \frac{44.4}{2} \text{ cm}$$

$$22 \text{ cm to } 22.2 \text{ cm}$$

$$r = 22 \text{ cm}, \quad dr = \delta r = 22.2 - 22 \\ = 0.2 \text{ cm}$$

$$dA = \pi d(r^2)$$

$$dA = 2\pi r dr$$

$$= 2\pi(22)(0.2)$$

$$= 27.467 \text{ cm}^2$$

$$\therefore \quad \text{Increase in area} = dA = 27.467 \text{ cm}^2 \quad \text{Ans.}$$



EXERCISE 3.2

Q.1 Evaluate the following indefinite integrals

(i) $\int (3x^2 - 2x + 1) dx$

(iii) $\int x (\sqrt{x} + 1) dx, (x > 0)$

(v) $\int (\sqrt{x} + 1)^2 dx (x > 0)$

(vii) $\int \frac{3x + 2}{\sqrt{x}} dx (x > 0)$

(ix) $\int \frac{(\sqrt{\theta} - 1)^2}{\sqrt{\theta}} d\theta (\theta > 0)$

(xi) $\int \frac{e^{2x} + e^x}{e^x} dx$

(ii) $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right) dx (x > 0)$

(iv) $\int (2x + 3)^{1/2} dx$

(vi) $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}} \right)^2 dx (x > 0)$

(viii) $\int \frac{\sqrt{y}(y + 1)}{y} dy, (y > 0)$

(x) $\int \frac{(1 - \sqrt{x})^2}{\sqrt{x}} dx, (x > 0)$

Solution:

(i) $\int (3x^2 - 2x + 1) dx$

$$= 3 \int x^2 dx - 2 \int x dx + \int dx$$

$$= \frac{3x^3}{3} - \frac{2x^2}{2} + x + c$$

$$= x^3 - x^2 + x + c \quad \text{Ans.}$$

(ii) $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right) dx (x > 0)$

$$= \int \sqrt{x} dx + \int \frac{dx}{\sqrt{x}}$$

$$= \int x^{1/2} dx + \int x^{-1/2} dx$$

$$= \frac{x^{3/2}}{\frac{3}{2}} + \frac{x^{1/2}}{\frac{1}{2}} + c$$

$$= \frac{2}{3} x^{3/2} + 2 \sqrt{x} + c \quad \text{Ans.}$$

(iii) $\int x (\sqrt{x} + 1) dx (x > 0)$

$$= \int (x \sqrt{x} + x) dx$$

$$= \int x \cdot x^{1/2} dx + \int x dx$$



$$\begin{aligned}
 &= \int x^{\frac{3}{2}} dx + \int x dx \\
 &= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{x^2}{2} + c \\
 &= \frac{2}{5} x^{\frac{5}{2}} + \frac{x^2}{2} + c \quad \text{Ans.}
 \end{aligned}$$

$$\begin{aligned}
 \text{(iv)} \quad &\int (2x + 3)^{\frac{1}{2}} dx \\
 &= \frac{1}{2} \int (2x + 3)^{\frac{1}{2}} 2dx \\
 &= \frac{1}{2} \frac{(2x + 3)^{\frac{3}{2}}}{\frac{3}{2}} + c \\
 &= \frac{(2x + 3)^{\frac{3}{2}}}{3} + c \quad \text{Ans}
 \end{aligned}$$

$$\begin{aligned}
 \text{(v)} \quad &\int (\sqrt{x} + 1)^2 dx \quad (x > 0) \\
 &= \int [(\sqrt{x})^2 + 2(\sqrt{x})(1) + (1)^2] dx \\
 &= \int x dx + 2 \int x^{\frac{1}{2}} dx + \int dx \\
 &= \frac{x^2}{2} + \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} + x + c \\
 &= \frac{x^2}{2} + \frac{2}{3} x^{\frac{3}{2}} + x + c \quad \text{Ans.}
 \end{aligned}$$

$$\begin{aligned}
 \text{(vi)} \quad &\int \left(\sqrt{x} - \frac{1}{x} \right)^2 dx \quad (x > 0) \\
 &= \int \left[(\sqrt{x})^2 - 2(\sqrt{x}) \left(\frac{1}{\sqrt{x}} \right) + \left(\frac{1}{\sqrt{x}} \right)^2 \right] dx \\
 &= \int \left(x - 2 + \frac{1}{x} \right) dx \\
 &= \int x dx - 2 \int dx + \int \frac{dx}{x}
 \end{aligned}$$



$$= \frac{x^2}{2} - 2x + \ln x + c \quad \text{Ans.}$$

$$(vii) \quad \int \frac{3x+2}{\sqrt{x}} dx \quad (x > 0)$$

$$= \int \left(\frac{3x}{\sqrt{x}} + \frac{2}{\sqrt{x}} \right) dx$$

$$= \int (3x^{1-\frac{1}{2}} + 2x^{-\frac{1}{2}}) dx$$

$$= 3 \int x^{\frac{1}{2}} dx + 2 \int x^{-\frac{1}{2}} dx$$

$$= \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= 2x^{\frac{3}{2}} + 4\sqrt{x} + c \quad \text{Ans.}$$

$$(viii) \quad \int \frac{\sqrt{y}(y+1)}{y} dy \quad (y > 0)$$

$$= \int \frac{\sqrt{y}(y+1)}{\sqrt{y} \cdot \sqrt{y}} dy$$

$$= \int \left(\frac{y}{\sqrt{y}} + \frac{1}{\sqrt{y}} \right) dy$$

$$= \int (y^{1-\frac{1}{2}} + y^{-\frac{1}{2}}) dy$$

$$= \int y^{\frac{1}{2}} dy + \int y^{-\frac{1}{2}} dy$$

$$= \frac{y^{\frac{3}{2}}}{\frac{3}{2}} + \frac{y^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= \frac{2}{3} y^{\frac{3}{2}} + 2y^{\frac{1}{2}} + c \quad \text{Ans.}$$

$$(ix) \quad \int \frac{(\sqrt{\theta}-1)^2}{\sqrt{\theta}} d\theta \quad (\theta > 0) \quad (\text{Guj. Board 2006})$$



$$\begin{aligned}
 &= \int \frac{(\sqrt{\theta})^2 - 2(\sqrt{\theta})(1) + (1)^2}{\sqrt{\theta}} d\theta \\
 &= \int \frac{\theta - 2\sqrt{\theta} + 1}{\sqrt{\theta}} d\theta \\
 &= \int \left(\frac{\theta}{\sqrt{\theta}} - \frac{2\sqrt{\theta}}{\sqrt{\theta}} + \frac{1}{\sqrt{\theta}} \right) d\theta \\
 &= \int (\theta^{1-\frac{1}{2}} - 2 + \theta^{-\frac{1}{2}}) d\theta \\
 &= \int \theta^{\frac{1}{2}} d\theta - 2 \int d\theta + \int \theta^{-\frac{1}{2}} d\theta \\
 &= \frac{\theta^{\frac{3}{2}}}{\frac{3}{2}} - 2\theta + \frac{\theta^{\frac{1}{2}}}{\frac{1}{2}} + c \\
 &= \frac{2}{3}\theta^{\frac{3}{2}} - 2\theta + 2\theta^{\frac{1}{2}} + c \quad \text{Ans.}
 \end{aligned}$$

(x) $\int \frac{(1 - \sqrt{x})^2}{\sqrt{x}} dx \quad (x > 0) \quad (\text{Lhr. Board 2006})$

$$\begin{aligned}
 &= \int \frac{(1)^2 - 2(1)(\sqrt{x}) + (\sqrt{x})^2}{\sqrt{x}} dx \\
 &= \int \frac{1 - 2\sqrt{x} + x}{\sqrt{x}} dx \\
 &= \int \left(\frac{1}{\sqrt{x}} - \frac{2\sqrt{x}}{\sqrt{x}} + \frac{x}{\sqrt{x}} \right) dx \\
 &= \int (x^{-\frac{1}{2}} - 2 + x^{1-\frac{1}{2}}) dx \\
 &= \int x^{-\frac{1}{2}} dx - 2 \int dx + \int x^{\frac{1}{2}} dx \\
 &= \frac{x^{\frac{1}{2}}}{\frac{1}{2}} - 2x + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c \\
 &= \frac{1}{2}x^{\frac{1}{2}} - 2x + \frac{2}{3}x^{\frac{3}{2}} + c \quad \text{Ans.}
 \end{aligned}$$

(xi) $\int \frac{e^{2x} + e^x}{e^x} dx$



$$\begin{aligned}
 &= \int \frac{e^x (e^x + 1)}{e^x} dx \\
 &= \int (e^x + 1) dx \\
 &= \int e^x dx + \int dx \\
 &= e^x + x + c \quad \text{Ans.}
 \end{aligned}$$

Q.2 Evaluate

$$(i) \int \frac{dx}{\sqrt{x+a} + \sqrt{x+b}} \quad \left(\begin{array}{l} x+a > 0 \\ x+b > 0 \end{array} \right)$$

$$(iii) \int \frac{dx}{\sqrt{x+a} + \sqrt{x}} \quad (x > 0, a > 0)$$

$$(v) \int \frac{(1+e^x)^3}{e^x} dx$$

$$(vii) \int \sqrt{1-\cos 2x} dx \quad (1 - \cos 2x > 0)$$

$$(ix) \int \sin^2 x dx$$

$$(xi) \int \frac{ax+b}{ax^2+2bx+c} dx$$

$$(xiii) \int \frac{\cos 2x - 1}{1 + \cos 2x} dx, (1 + \cos 2x \neq 0)$$

$$(ii) \int \frac{1-x^2}{1+x^2} dx$$

$$(iv) \int (a-2x)^{\frac{3}{2}} dx$$

$$(vi) \int \sin(a+b)x dx$$

$$(viii) \int (\ln x) \times \frac{1}{x} dx \quad (x > 0)$$

$$(x) \int \frac{1}{1+\cos x} dx \quad \left(-\frac{\pi}{2} < x < \frac{\pi}{2} \right)$$

$$(xii) \int \cos 3x \sin 2x dx$$

$$(xiv) \int \tan^2 x dx \quad (\text{Lhr. Board 2011})$$

Solution:

$$\begin{aligned}
 (i) & \int \frac{dx}{\sqrt{x+a} + \sqrt{x+b}} \quad \left(\begin{array}{l} x+a > 0 \\ x+b > 0 \end{array} \right) \\
 &= \int \frac{dx}{\sqrt{x+a} + \sqrt{x+b}} \times \frac{\sqrt{x+a} - \sqrt{x+b}}{\sqrt{x+a} - \sqrt{x+b}} \\
 &= \int \frac{\sqrt{x+a} - \sqrt{x+b}}{\sqrt{(x+a)^2} - \sqrt{(x+b)^2}} dx \\
 &= \int \frac{\sqrt{x+a} - \sqrt{x+b}}{x+a - (x+b)} dx \\
 &= \int \frac{(x+a)^{\frac{1}{2}} - (x+b)^{\frac{1}{2}}}{x+a-x-b} dx
 \end{aligned}$$



$$\begin{aligned}
 &= \frac{1}{a-b} \int [(x+a)^{1/2} - (x-b)^{1/2}] dx \\
 &= \frac{1}{a-b} \left[\int (x+a)^{1/2} dx - \int (x-b)^{1/2} dx \right] \\
 &= \frac{1}{a-b} \left[\frac{(x+a)^{3/2}}{\frac{3}{2}} - \frac{(x-b)^{3/2}}{\frac{3}{2}} \right] + c \\
 &= \frac{2}{3(a-b)} [(x+b)^{3/2} - (x-b)^{3/2}] + c \quad \text{Ans.}
 \end{aligned}$$

(ii) $\int \frac{1-x^2}{1+x^2} dx$ (Lhr. Board 2008)

$$\begin{aligned}
 &= \int \left(-1 + \frac{2}{1+x^2} \right) dx \\
 &= -\int dx + 2 \int \frac{dx}{1+x^2} \\
 &= -x + 2 \tan^{-1} x + c \quad \text{Ans.}
 \end{aligned}$$

(iii) $\int \frac{dx}{\sqrt{x+a} + \sqrt{x}} \quad (x > 0, a > 0)$

$$\begin{aligned}
 &= \int \frac{dx}{\sqrt{x+a} + \sqrt{x}} \times \frac{\sqrt{x+a} - \sqrt{x}}{\sqrt{x+a} - \sqrt{x}} \\
 &= \int \frac{\sqrt{x+a} - \sqrt{x}}{(\sqrt{x+a})^2 - (\sqrt{x})^2} dx \\
 &= \int \frac{(x+a)^{1/2} - x^{1/2}}{x+a-x} dx \\
 &= \frac{1}{a} \int [(x+a)^{1/2} - x^{1/2}] dx \\
 &= \frac{1}{a} \left[\int (x+a)^{1/2} dx - \int x^{1/2} dx \right]
 \end{aligned}$$



$$\begin{aligned}
 &= \frac{1}{a} \left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right] + c \\
 &= \frac{2}{3a} \left[(x+a)^{\frac{3}{2}} - x^{\frac{3}{2}} \right] + c \quad \text{Ans.}
 \end{aligned}$$

$$\begin{aligned}
 \text{(iv)} \quad &\int (a-2x)^{3/2} dx \\
 &= \frac{-1}{2} \int (a-2x)^{3/2} \cdot 2 dx \\
 &= \frac{-1}{2} \frac{(a-2x)^{5/2}}{5/2} + c \\
 &= \frac{-1}{5} (a-2x)^{5/2} + c \quad \text{Ans.}
 \end{aligned}$$

$$\begin{aligned}
 \text{(v)} \quad &\int \frac{(1+e^x)^3}{e^x} dx \\
 &= \int \frac{(1)^3 + 3(1)^2(e^x) + 3(1)(e^x)^2 + (e^x)^3}{e^x} dx \\
 &= \int \frac{1 + 3e^x + 3e^{2x} + e^{3x}}{e^x} dx \\
 &= \int \left(\frac{1}{e^x} + \frac{3e^x}{e^x} + \frac{3e^{2x}}{e^x} + \frac{e^{3x}}{e^x} \right) dx \\
 &= \int (e^{-x} + 3 + 3e^x + e^{2x}) dx \\
 &= -\int e^{-x} dx + 3 \int dx + 3 \int e^x dx + \frac{1}{2} \int e^{2x} \cdot 2 dx + \frac{1}{2} \int e^{2x} \cdot 2 dx \\
 &= -e^{-x} + 3x + 3e^x + \frac{1}{2} e^{2x} + c \quad \text{Ans.}
 \end{aligned}$$

$$\begin{aligned}
 \text{(vi)} \quad &\int \sin(a+b)x dx \\
 &= \frac{1}{a+b} \int \sin(a+b)x \cdot (a+b) dx
 \end{aligned}$$



$$= \frac{1}{a+b} \cdot -\cos(a+b)x + c$$

$$= -\frac{1}{a+b} \cos(a+b)x + c \quad \text{Ans.}$$

(vii) $\int \sqrt{1 - \cos 2x} \, dx \quad (1 - \cos 2x > 0) \quad (\text{Lhr. Board 2008, 2009})$

$$= \int \sqrt{2\sin^2 x} \, dx \quad \left(\begin{array}{l} \because \cos 2x = 1 - 2\sin^2 x \\ 2\sin^2 x = 1 - \cos 2x \end{array} \right)$$

$$= \sqrt{2} \int \sin x \, dx$$

$$= \sqrt{2} (-\cos x) + c$$

$$= -\sqrt{2} \cos x + c \quad \text{Ans.}$$

(viii) $\int (\ln x) \times \frac{1}{x} \, dx \quad (x > 0)$

$$= \frac{(\ln x)^2}{2} + c \quad \text{Ans.}$$

(ix) $\int \sin^2 x \, dx$

$$= \int \frac{1 - \cos 2x}{2} \, dx \quad \left(\begin{array}{l} \because \cos 2x = 1 - 2\sin^2 x \\ 2\sin^2 x = 1 - \cos 2x \\ \sin^2 x = \frac{1 - \cos 2x}{2} \end{array} \right)$$

$$= \frac{1}{2} \int (1 - \cos 2x) \, dx$$

$$= \frac{1}{2} \int dx - \frac{1}{2 \cdot 2} \int \cos 2x \cdot 2 \, dx$$

$$= \frac{1}{2} x - \frac{1}{4} \sin 2x + c \quad \text{Ans.}$$

(x) $\int \frac{1}{1 + \cos x} \, dx \quad \left(-\frac{\pi}{2} < x < \frac{\pi}{2} \right)$



$$\begin{aligned} &= \int \frac{1}{2\cos^2 \frac{x}{2}} dx \quad \left(\begin{array}{l} \because \cos 2x = 2\cos^2 x - 1 \\ 2\cos^2 x = 1 + \cos 2x \\ 2\cos^2 \frac{x}{2} = 1 + \cos x \end{array} \right) \\ &= \int \sec^2 \frac{x}{2} \cdot \frac{1}{2} dx \\ &= \tan \frac{x}{2} + c \quad \text{Ans.} \end{aligned}$$

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$$\begin{aligned}
 \text{(xi)} \quad & \int \frac{ax + b}{ax^2 + 2bx + c} dx \\
 &= \frac{1}{2} \int \frac{2ax + 2b}{ax^2 + 2bx + c} dx \\
 &= \frac{1}{2} \ln |(ax^2 + 2bx + c)| + c_1 \quad \text{Ans.}
 \end{aligned}$$

$$\begin{aligned}
 \text{(xii)} \quad & \int \cos 3x \sin 2x dx \\
 &= \frac{1}{2} \int 2 \cos 3x \sin 2x dx \\
 &= \frac{1}{2} \int [\sin (3x + 2x) - \sin (3x - 2x)] dx \\
 & \qquad \qquad \qquad \because 2 \cos \alpha \sin \beta = \sin (\alpha + \beta) - \sin (\alpha - \beta) \\
 &= \frac{1}{2} \int (\sin 5x - \sin x) dx \\
 &= \frac{1}{2} \left[\int \sin 5x dx - \int \sin x dx \right] \\
 &= \frac{1}{2} \left[\frac{1}{5} \int \sin 5x \cdot 5 dx - \int \sin x dx \right] \\
 &= \frac{1}{2} \left(\frac{-\cos 5x}{5} + \cos x \right) + c \\
 &= \frac{-1}{2} \left(\frac{\cos 5x}{5} - \cos x \right) + c \quad \text{Ans.}
 \end{aligned}$$

$$\begin{aligned}
 \text{(xiii)} \quad & \int \frac{\cos 2x - 1}{1 + \cos 2x} dx, \quad (1 + \cos 2x \neq 0) \\
 &= - \int \frac{1 - \cos 2x}{1 + \cos 2x} dx \\
 &= - \int \frac{2 \sin^2 x}{2 \cos^2 x} dx \\
 &= - \int \tan^2 x dx
 \end{aligned}$$



$$\begin{aligned} &= -\int (\sec^2 x - 1) dx \\ &= -\int \sec^2 x dx + \int dx \\ &= x - \tan x + c \quad \text{Ans.} \end{aligned}$$

(xiv) $\int \tan^2 x dx$ (Guj. Board 2005, 2007) (Lhr. Board 2011)

$$\begin{aligned} &= \int (\sec^2 x - 1) dx \\ &= \int \sec^2 x dx - \int dx \\ &= \tan x - x + c \quad \text{Ans.} \end{aligned}$$



EXERCISE 3.3

Evaluate the following integrals.

Q.1 $\int \frac{-2x}{\sqrt{4-x^2}} dx$

Solution:

$$\begin{aligned} & \int \frac{-2x}{\sqrt{4-x^2}} dx \\ &= \int (4-x^2)^{-1/2} - 2x dx \\ &= \frac{(4-x^2)^{1/2}}{\frac{1}{2}} + c \\ &= 2\sqrt{4-x^2} + c \quad \text{Ans.} \end{aligned}$$

Q.2 $\int \frac{dx}{x^2+4x+13}$

Solution:

$$\int \frac{dx}{x^2+4x+13}$$



$$\begin{aligned}
 &= \int \frac{dx}{x^2 + 4x + 4 - 4 + 13} \\
 &= \int \frac{dx}{(x+2)^2 + 9} \\
 &= \int \frac{dx}{(x+2)^2 + (3)^2} \\
 &= \frac{1}{3} \tan^{-1} \left(\frac{x+2}{3} \right) + c \quad \text{Ans.}
 \end{aligned}$$

$$\therefore \int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$$

Q.3 $\int \frac{x^2}{4+x^2} dx$

Solution:

$$\begin{aligned}
 &\int \frac{x^2}{4+x^2} dx \\
 &= \int \left(1 - \frac{4}{4+x^2} \right) dx \\
 &= \int dx - 4 \int \frac{dx}{(2)^2 + x^2} \\
 &= x - 2 \tan^{-1} \left(\frac{x}{2} \right) + c \quad \text{Ans.}
 \end{aligned}$$

$$4+x^2 \sqrt{\frac{1}{x^2}} = \frac{-x^2 \pm 4}{-4}$$

$$= x - 4 \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) + c$$

Q.4 $\int \frac{1}{x \ln x} dx$

Solution:

$$\begin{aligned}
 &\int \frac{1}{x \ln x} dx \\
 &= \int \frac{1/x}{\ln x} dx \\
 &= \ln (\ln x) + c \quad \text{Ans.}
 \end{aligned}$$

$$\therefore \int [f(x)]^{-1} \cdot f'(x) dx = \ln [f(x)] + c$$

$$f(x) = \ln x$$

$$f'(x) = \frac{1}{x}$$

Q.5 $\int \frac{e^x}{e^x+3} dx$

Solution:

$$\begin{aligned}
 &\int \frac{e^x}{e^x+3} dx \\
 &= \ln (e^x + 3) + c \quad \text{Ans.}
 \end{aligned}$$



Q.6 $\int \frac{x+b}{(x^2+2bx+c)^{1/2}} dx$

Solution:

$$\int \frac{x+b}{(x^2+2bx+c)^{1/2}} dx$$

$$= \frac{1}{2} \int (x^2+2bx+c)^{-1/2} \cdot 2(x+b) dx$$

$$= \frac{1}{2} \int (x^2+2bx+c)^{-1/2} (2x+2b) dx$$

$$= \frac{1}{2} \frac{(x^2+2bx+c)^{1/2}}{\frac{1}{2}} + c$$

$$= \sqrt{x^2+2bx+c} + c \quad \text{Ans.}$$

$$\begin{aligned} f(x) &= x^2 + 2bx + c \\ f'(x) &= 2x + 2b \\ f'(x) &= 2(x+b) \end{aligned}$$

$$\therefore \int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c$$

Q.7 $\int \frac{\sec^2 x}{\sqrt{\tan x}} dx$ (Lhr. Board 2005, 2008)

Solution:

$$\int \frac{\sec^2 x}{\sqrt{\tan x}} dx$$

$$= \int (\tan x)^{-1/2} \cdot \sec^2 x dx$$

$$= \frac{(\tan x)^{1/2}}{\frac{1}{2}} + c$$

$$= 2\sqrt{\tan x} + c \quad \text{Ans.}$$

$$\begin{aligned} f(x) &= \tan x \\ f'(x) &= \sec^2 x \end{aligned}$$

$$\therefore \int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c$$

Q.8 (a) Show that $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln(x + \sqrt{x^2-a^2}) + c$

(b) Show that $\int \sqrt{a^2-x^2} dx = \frac{a^2}{2} \sin^{-1} x + \frac{x}{a} \sqrt{a^2-x^2} + c$

Solution:

(a) Taking

$$\int \frac{dx}{\sqrt{(x^2-a^2)}}$$



Put

$$x = a \sec \theta \quad \Rightarrow \quad \sec \theta = \frac{x}{a}$$

$$dx = a \sec \theta \tan \theta \, d\theta$$

$$= \int \frac{a \sec \theta \tan \theta \, d\theta}{\sqrt{a^2 \sec^2 \theta - a^2}}$$

$$= \int \frac{a \sec \theta \tan \theta}{\sqrt{a^2(\sec^2 \theta - 1)}} \, d\theta$$

$$= \int \frac{a \sec \theta \tan \theta}{a \sqrt{\tan^2 \theta}} \, d\theta$$

$$= \int \frac{a \sec \theta \tan \theta}{\tan \theta} \, d\theta$$

$$= \int \sec \theta \, d\theta$$

$$= \ln (\sec \theta + \tan \theta) + c_1$$

$$= \ln \left(\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right) + c_1$$

$$= \ln \left(\frac{x + \sqrt{x^2 - a^2}}{a} \right) + c_1$$

$$= \ln (x + \sqrt{x^2 - a^2}) - \ln (a) + c_1$$

$$= \ln (x + \sqrt{x^2 - a^2}) + c \quad \text{where } c = -\ln a + c_1$$

Hence proved.

(b) Taking

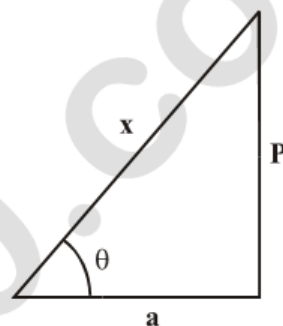
$$\int \sqrt{a^2 - x^2} \, dx$$

$$\text{Put } x = a \sin \theta \quad \Rightarrow \quad \sin \theta = \frac{x}{a} \quad \Rightarrow \quad \theta = \sin^{-1} \left(\frac{x}{a} \right)$$

$$dx = a \cos \theta \, d\theta \quad \Rightarrow \quad \theta = \sin^{-1} \frac{x}{a}$$

$$= \int \sqrt{a^2 - a^2 \sin^2 \theta} \cdot a \cos \theta \, d\theta$$

$$= \int \sqrt{a^2(1 - \sin^2 \theta)} \cdot a \cos \theta \, d\theta$$



$$x^2 = a^2 + P^2$$

$$P^2 = x^2 - a^2$$

$$P = \sqrt{x^2 - a^2}$$

$$\left| \begin{array}{l} \tan \theta = \frac{P}{a} \\ = \frac{\sqrt{x^2 - a^2}}{a} \end{array} \right.$$



$$\begin{aligned}
 &= \int a \sqrt{\cos^2 \theta} \cdot a \cos \theta \, d\theta \\
 &= a^2 \int \cos \theta \cdot \cos \theta \, d\theta \\
 &= a^2 \int \cos^2 \theta \, d\theta \\
 &= a^2 \int \frac{1 + \cos 2\theta}{2} \, d\theta \quad \left(\begin{array}{l} \because \cos 2\theta = 2\cos^2 \theta - 1 \\ 2\cos^2 \theta = 1 + \cos 2\theta \\ \cos^2 \theta = \frac{1 + \cos 2\theta}{2} \end{array} \right) \\
 &= \frac{a^2}{2} \int (1 + \cos 2\theta) \, d\theta
 \end{aligned}$$

$$= \frac{a^2}{2} \int d\theta + \frac{a^2}{2} \int \cos 2\theta \, d\theta$$

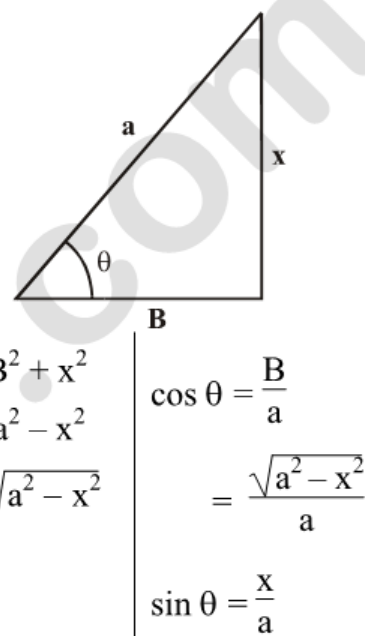
$$= \frac{a^2}{2} \theta + \frac{a^2}{2} \cdot \frac{\sin 2\theta}{2} + c$$

$$= \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{a^2}{4} \cdot 2 \sin \theta \cos \theta + c$$

$$= \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{a^2}{2} \cdot \frac{x}{a} \cdot \frac{\sqrt{a^2 - x^2}}{a} + c$$

$$= \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2} + c$$

Hence proved



Evaluate the following integrals

Q.9 $\int \frac{dx}{(1+x^2)^{3/2}}$

Solution:

$$\int \frac{dx}{(1+x^2)^{3/2}}$$

Put $x = \tan \theta$

$$dx = \sec^2 \theta \, d\theta$$

$$= \int \frac{\sec^2 \theta \, d\theta}{(1 + \tan^2 \theta)^{3/2}} = \int \frac{\sec^2 \theta}{(\sec^2 \theta)^{3/2}} \, d\theta$$



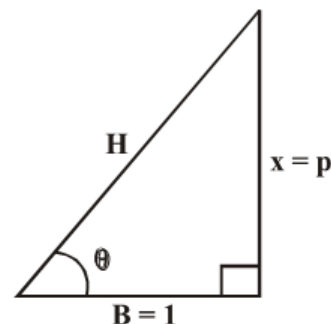
$$= \int \frac{\sec^2 \theta}{\sec^3 \theta} d\theta$$

$$= \int \frac{d\theta}{\sec \theta} d\theta$$

$$= \int \cos \theta d\theta$$

$$= \sin \theta + c$$

$$\begin{aligned} \therefore \sin \theta &= \frac{P}{H} \\ \sin \theta &= \frac{x}{\sqrt{1+x^2}} \end{aligned}$$



$$H^2 = 1 + x^2$$

$$H = \sqrt{1 + x^2}$$

$$= \frac{x}{\sqrt{1+x^2}} + c$$

Ans.

Q.10 $\int \frac{1}{(1+x^2) \tan^{-1} x} dx$

Solution:

$$\int \frac{1}{(1+x^2) \tan^{-1} x} dx$$

$$= \int \frac{1}{\frac{1+x^2}{\tan^{-1} x}} dx$$

$$= \ln (\tan^{-1} x) + c$$

Ans.

$$\therefore f(x) = \tan^{-1} x$$

$$\therefore f'(x) = \frac{1}{1+x^2}$$

Q.11 $\int \sqrt{\frac{1+x}{1-x}} dx$

Solution:

$$\int \sqrt{\frac{1+x}{1-x}} dx$$

$$= \int \sqrt{\frac{1+x}{1-x} \times \frac{1+x}{1+x}} dx$$

$$= \int \sqrt{\frac{(1+x)^2}{1-x^2}} dx$$

$$= \int \frac{1+x}{\sqrt{1-x^2}} dx$$

 $\therefore$ L.C.M Breaking

$$\begin{aligned}
&= \int \left(\frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}} \right) dx \\
&= \int \frac{dx}{\sqrt{1-x^2}} - \frac{1}{2} \int (1-x^2)^{-1/2} \cdot -2x dx \\
&= \sin^{-1}x - \frac{1}{2} \frac{(1-x^2)^{1/2}}{\frac{1}{2}} + c \\
&= \sin^{-1}x - \sqrt{1-x^2} + c \quad \text{Ans.}
\end{aligned}$$

Q.12 $\int \frac{\sin \theta}{1 + \cos^2 \theta} d\theta$

Solution:

$$\int \frac{\sin \theta}{1 + \cos^2 \theta} d\theta$$

Put

$$\cos \theta = t$$

$$-\sin \theta d\theta = dt$$

$$d\theta = \frac{dt}{-\sin \theta}$$

$$= \int \frac{\sin \theta}{1 + t^2} \times \frac{dt}{-\sin \theta}$$

$$= - \int \frac{dt}{1 + t^2}$$

$$= - \tan^{-1}(t) + c$$

$$= - \tan^{-1}(\cos \theta) + c \quad \text{Ans.} \quad \because t = \cos \theta$$

Q.13 $\int \frac{ax}{\sqrt{a^2 - x^4}} dx$

Solution:

$$\int \frac{ax}{\sqrt{a^2 - x^4}} dx$$



$$= \int \frac{ax}{\sqrt{a^2 - (x^2)^2}} dx$$

Put

$$x^2 = t$$

$$2x dx = dt$$

$$dx = \frac{dt}{2x}$$

$$= \int \frac{ax}{\sqrt{a^2 - t^2}} \times \frac{dt}{2x}$$

$$= \frac{a}{2} \int \frac{dt}{\sqrt{a^2 - t^2}}$$

$$= \frac{a}{2} \sin^{-1} \left(\frac{t}{a} \right) + c$$

$$= \frac{a}{2} \sin^{-1} \left(\frac{x^2}{a} \right) + c$$

Ans. $\therefore t = x^2$

Q.14 $\int \frac{dx}{\sqrt{7 - 6x - x^2}}$

Solution:

$$\int \frac{dx}{\sqrt{7 - 6x - x^2}}$$

$$= \int \frac{dx}{\sqrt{-(x^2 + 6x - 7)}}$$

$$= \int \frac{dx}{\sqrt{-(x^2 + 6x + 9 - 9 - 7)}}$$

$$= \int \frac{dx}{\sqrt{-(x + 3)^2 - 16}}$$



$$\begin{aligned}
 &= \int \frac{dx}{\sqrt{16 - (x+3)^2}} \\
 &= \int \frac{dx}{\sqrt{(4)^2 - (x+3)^2}} \quad \because \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + c \\
 &= \sin^{-1} \left(\frac{x+3}{4} \right) + C \quad \text{Ans.}
 \end{aligned}$$

Q.15 $\int \frac{\cos x}{\sin x \ln \sin x} dx$ (Guj. Board 2008)

Solution:

$$\int \frac{\cos x}{\sin x \ln \sin x} dx$$

Put

$$\ln \sin x = t$$

$$\frac{1}{\sin x} \cos x dx = dt$$

$$dx = \frac{\sin x}{\cos x} dt$$

$$= \int \frac{\cos x}{\sin x t} \times \frac{\sin x}{\cos x} dt$$

$$= \int \frac{dt}{t}$$

$$= \ln t + c$$

$$= \ln (\ln \sin x) + c \quad \text{Ans.} \quad \because t = \ln \sin x$$

Q.16 $\int \cos x \left(\frac{\ln \sin x}{\sin x} \right) dx$ (Lhr. Board 2005)

Solution:

$$\int \cos x \left(\frac{\ln \sin x}{\sin x} \right) dx$$

Put

$$\ln \sin x = t$$



$$\begin{aligned}
 \frac{1}{\sin x} \cos x \cdot dx &= dt \\
 dx &= \frac{\sin x}{\cos x} \\
 &= \int \cos x \left(\frac{t}{\sin x} \right) \cdot \frac{\sin x}{\cos x} dt \\
 &= \int t dt \\
 &= \frac{t^2}{2} + c \\
 &= \frac{(\ln \sin x)^2}{2} + c \quad \text{Ans.} \quad \because t = \ln \sin x
 \end{aligned}$$

Q.17 $\int \frac{x dx}{4 + 2x + x^2}$

Solution:

Formula used

$$\because \int [f(x)^{-1}] f'(x) dx = \ln f(x) + c$$

$$\because \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$$

$$\begin{aligned}
 &\int \frac{x dx}{4 + 2x + x^2} \\
 &= \frac{1}{2} \int \frac{2x + 2 - 2}{4 + 2x + x^2} dx \\
 &= \frac{1}{2} \int \left(\frac{2x + 2}{4 + 2x + x^2} - \frac{2}{4 + 2x + x^2} \right) dx \\
 &= \frac{1}{2} \int \frac{2x + 2}{4 + 2x + x^2} dx - \frac{2}{2} \int \frac{dx}{4 + 2x + x^2} \\
 &= \frac{1}{2} \ln (x^2 + 2x + 4) - \int \frac{dx}{x^2 + 2x + 1 - 1 + 4} \\
 &= \frac{1}{2} \ln (x^2 + 2x + 4) - \int \frac{dx}{(x + 1)^2 + 3} \\
 &= \frac{1}{2} \ln (x^2 + 2x + 4) - \int \frac{dx}{(x + 1)^2 + (\sqrt{3})^2} \\
 &= \frac{1}{2} \ln (x^2 + 2x + 4) - \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{x + 1}{\sqrt{3}} \right) + c \quad \text{Ans.}
 \end{aligned}$$



Q.18 $\int \frac{x}{x^4 + 2x^2 + 5} dx$

Solution:

$$\begin{aligned} & \int \frac{x}{x^4 + 2x^2 + 5} dx \\ &= \int \frac{x}{(x^2)^2 + 2x^2 + 5} dx \\ \text{Put } & x^2 = t \\ & 2x dx = dt \\ & dx = \frac{dt}{2x} \\ &= \int \frac{x}{t^2 + 2t + 5} \times \frac{dt}{2x} \\ &= \frac{1}{2} \int \frac{dt}{t^2 + 2t + 1 - 1 + 5} \\ &= \frac{1}{2} \int \frac{dt}{(t+1)^2 + 4} \\ &= \frac{1}{2} \int \frac{dt}{(t+1)^2 + (2)^2} \\ &= \frac{1}{2} \cdot \frac{1}{2} \tan^{-1} \left(\frac{t+1}{2} \right) + c \\ &= \frac{1}{4} \tan^{-1} \left(\frac{x^2 + 1}{2} \right) + c \quad \text{Ans.} \end{aligned}$$

$$\because t = x^2$$

Q.19 $\int \left[\cos \left(\sqrt{x} - \frac{x}{2} \right) \right] \times \left(\frac{1}{\sqrt{x}} - 1 \right) dx$

Solution:

$$\begin{aligned} & \int \left[\cos \left(\sqrt{x} - \frac{x}{2} \right) \right] \times \left(\frac{1}{\sqrt{x}} - 1 \right) dx \\ &= 2 \int \cos \left(\sqrt{x} - \frac{x}{2} \right) \times \frac{1}{2} \left(\frac{1}{\sqrt{x}} - 1 \right) dx \end{aligned}$$

$$\begin{aligned} \because f(x) &= \sqrt{x} - \frac{x}{2} \\ f'(x) &= \frac{1}{2\sqrt{x}} - \frac{1}{2} \\ f'(x) &= \frac{1}{2} \left(\frac{1}{\sqrt{x}} - 1 \right) \end{aligned}$$



$$= 2 \int \cos \left(\sqrt{x} - \frac{x}{2} \right) \times \left(\frac{1}{2\sqrt{x}} - \frac{1}{2} \right) dx$$

$$= 2 \sin \left(\sqrt{x} - \frac{x}{2} \right) + c \quad \text{Ans.}$$

Q.20 $\int \frac{x+2}{\sqrt{x+3}} dx$

Solution:

$$\int \frac{x+2}{\sqrt{x+3}} dx$$

Put

$$\sqrt{x+3} = t$$

$$x+3 = t^2 \quad \Rightarrow \quad x = t^2 - 3$$

$$dx = 2t dt$$

$$= \int \frac{t^2 - 3 + 2}{t} \times 2t dt$$

$$= 2 \int (t^2 - 1) dt$$

$$= 2 \int t^2 dt - 2 \int dt$$

$$= \frac{2t^3}{3} - 2t + c$$

$$= \frac{2}{3} (x+3)^{\frac{3}{2}} - 2\sqrt{x+3} + c \quad \text{Ans.} \quad \because t = \sqrt{x+3}$$

Q.21 $\int \frac{\sqrt{2}}{\sin x + \cos x} dx$ (Lhr. Board 2008)

Solution:

$$\int \frac{\sqrt{2}}{\sin x + \cos x} dx$$

$$= \int \frac{1}{\frac{1}{\sqrt{2}} (\sin x + \cos x)} dx$$



$$\begin{aligned}
 &= \int \frac{dx}{\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x} \\
 &= \int \frac{dx}{\cos x \cos \frac{\pi}{4} + \sin x \sin \frac{\pi}{4}} \quad \because \cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \\
 &= \int \frac{dx}{\cos \left(x - \frac{\pi}{4}\right)} \\
 &= \int \sec \left(x - \frac{\pi}{4}\right) dx \\
 &= \ln \left| \sec \left(x - \frac{\pi}{4}\right) + \tan \left(x - \frac{\pi}{4}\right) \right| + c \quad \text{Ans.}
 \end{aligned}$$

Q.22 $\int \frac{dx}{\frac{1}{2} \sin x + \frac{\sqrt{3}}{2} \cos x}$

Solution:

$$\begin{aligned}
 &\int \frac{dx}{\frac{1}{2} \sin x + \frac{\sqrt{3}}{2} \cos x} \\
 &= \int \frac{dx}{\sin x \cos \frac{\pi}{3} + \cos x \sin \frac{\pi}{3}} \quad \because \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \\
 &= \int \frac{dx}{\sin \left(x + \frac{\pi}{3}\right)} \\
 &= \int \operatorname{cosec} \left(x + \frac{\pi}{3}\right) dx \\
 &= \ln \left| \operatorname{cosec} \left(x + \frac{\pi}{3}\right) - \cot \left(x + \frac{\pi}{3}\right) \right| + c \quad \text{Ans.}
 \end{aligned}$$



EXERCISE 3.4

Q.2 Evaluate the following integrals by parts add a word representing all the functions are defined.

- | | | |
|------------------------------------|--|---|
| (i) $\int x \sin x \, dx$ | (ii) $\int \ln x \, dx$ | (iii) $\int x / \ln x \, dx$ |
| (iv) $\int x^2 \ln x \, dx$ | (v) $\int x^3 \ln x \, dx$ | (vi) $\int x^4 \ln x \, dx$ |
| (vii) $\int \tan^{-1} x \, dx$ | (viii) $\int x^2 \sin x \, dx$ | (ix) $\int x^2 \tan^{-1} x \, dx$ |
| (x) $\int x \tan^{-1} x \, dx$ | (xi) $\int x^3 \tan^{-1} x \, dx$ | (xii) $\int x^3 \cos x \, dx$ |
| (xiii) $\int \sin^{-1} x \, dx$ | (xiv) $\int x \sin^{-1} x \, dx$ | (xv) $\int e^x \sin x \cos x \, dx$ |
| (xvi) $\int x \sin x \cos x \, dx$ | (xvii) $\int x \cos^2 x \, dx$ | (xviii) $\int x \sin^2 x \, dx$ |
| (xix) $\int (\ln x)^2 \, dx$ | (xx) $\int \ln(\tan x) \sec^2 x \, dx$ | (xxi) $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} \, dx$ |

Solution:

(i) $\int x \sin x \, dx$

$$= x \int \sin x \, dx - \int \left[\int \sin x \, dx \cdot \frac{d}{dx}(x) \right] dx$$

$$= -x \cos x - \int -\cos x \cdot 1 \, dx$$

$$= -x \cos x + \int \cos x \, dx$$

$$= -x \cos x + \sin x + c$$

$$= \sin x - x \cos x + c$$

Ans.

(ii) $\int \ln x \, dx$ (Guj. Board 2006) (Lhr. Board 2006)

$$= \int \ln x \cdot 1 \, dx$$

$\because \ln x \rightarrow 1^{\text{st}} \text{ function}$
 $1 \rightarrow 2^{\text{nd}} \text{ function}$

$\because x \rightarrow 1^{\text{st}} \text{ function}$
 $\sin x \rightarrow 2^{\text{nd}} \text{ function}$



$$= \ln x \int dx - \int \left[\int dx \cdot \frac{d}{dx} (\ln x) \right] dx$$

$$= \ln x \cdot x - \int x \cdot \frac{1}{x} dx$$

$$= x \ln x - \int dx$$

$$= x \ln x - x + c \quad \text{Ans.}$$

(iii) $\int x \ln x dx$ (Lhr. Board 2006, 2007, 2008)

$$= \ln x \int x dx - \int \left[\int x dx \cdot \frac{d}{dx} (\ln x) \right] dx$$

$$= \ln x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} dx$$

$$= \frac{x^2 \ln x}{2} - \frac{1}{2} \int x dx$$

$$= \frac{x^2 \ln x}{2} - \frac{1}{2} \cdot \frac{x^2}{2} + c$$

$$= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + c$$

$$= \frac{1}{2} x^2 \left(\ln x - \frac{1}{2} \right) + c \quad \text{Ans.}$$

$\therefore \ln x$	$\rightarrow$	1 st function
x	$\rightarrow$	2 nd function

(iv) $\int x^2 \ln x dx$

$$= \ln x \int x^2 dx - \int \left[\int x^2 dx \cdot \frac{d}{dx} (\ln x) \right] dx$$

$$= \ln x \cdot \frac{x^3}{3} - \int \frac{x^3}{3} \cdot \frac{1}{x} dx$$

$$= \frac{x^3 \ln x}{3} - \frac{1}{3} \int x^2 dx$$

$$= \frac{x^3 \ln x}{3} - \frac{1}{3} \cdot \frac{x^3}{3} + c$$

$$= \frac{x^3}{3} \left(\ln x - \frac{1}{3} \right) + c \quad \text{Ans.}$$

$\therefore \ln x$	$\rightarrow$	1 st function
x^2	$\rightarrow$	2 nd function

(v) $\int x^3 \ln x dx$ (Lhr. Board 2007)

$\therefore \ln x$	$\rightarrow$	1 st function
x^3	$\rightarrow$	2 nd function



$$\begin{aligned}
&= \ln x \int x^3 dx - \int \left[\int x^3 dx \cdot \frac{d}{dx} (\ln x) \right] dx \\
&= \ln x \cdot \frac{x^4}{4} - \int \frac{x^4}{4} \cdot \frac{1}{x} dx \\
&= \frac{x^4 \ln x}{4} - \frac{1}{4} \int x^3 dx \\
&= \frac{x^4 \ln x}{4} - \frac{1}{4} \cdot \frac{x^4}{4} + c \\
&= \frac{x^4}{4} \left(\ln x - \frac{1}{4} \right) + c \quad \text{Ans.}
\end{aligned}$$

(vi) $\int x^4 \ln x dx$

$$\begin{aligned}
&= \ln x \int x^4 dx - \int \left[\int x^4 dx \cdot \frac{d}{dx} (\ln x) \right] dx \\
&= \ln x \cdot \frac{x^5}{5} - \int \frac{x^5}{5} \cdot \frac{1}{x} dx \\
&= \frac{x^5 \ln x}{5} - \frac{1}{5} \int x^4 dx \\
&= \frac{x^5 \ln x}{5} - \frac{1}{5} \cdot \frac{x^5}{5} + c \\
&= \frac{x^5}{5} \left(\ln x - \frac{1}{5} \right) + c \quad \text{Ans.}
\end{aligned}$$

$\because \ln x \rightarrow 1^{\text{st}} \text{ function}$
 $x^4 \rightarrow 2^{\text{nd}} \text{ function}$

(vii) $\int \tan^{-1} x dx$ (Lhr. Board 2011) (Guj. Board 2008)

$$\begin{aligned}
&= \int \tan^{-1} x \cdot 1 dx \\
&= \tan^{-1} x \int dx - \int \left[\int dx \cdot \frac{d}{dx} (\tan^{-1} x) \right] dx \\
&= \tan^{-1} x \cdot x - \int x \cdot \frac{1}{1+x^2} dx \\
&= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} dx
\end{aligned}$$

$\because \tan^{-1} x \rightarrow 1^{\text{st}} \text{ function}$
 $x \rightarrow 2^{\text{nd}} \text{ function}$

$\because f(x) = 1 + x^2$
 $f'(x) = 2x$



$$= \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + c \quad \text{Ans.}$$

(viii) $\int x^2 \sin x \, dx$

$$= x^2 \int \sin x \, dx - \int \left[\sin x \, dx \cdot \frac{d}{dx} (x^2) \right] dx$$

$$\begin{array}{l} \because x^2 \rightarrow 1^{\text{st}} \text{ function} \\ \sin x \rightarrow 2^{\text{nd}} \text{ function} \end{array}$$

$$= x^2 (-\cos x) - \int -\cos x \cdot 2x \, dx$$

$$= -x^2 \cos x + 2 \int x \cos x \, dx$$

$$= -x^2 \cos x + 2 \left[x \sin x \, dx - \int \left(\int \sin x \, dx \cdot \frac{d}{dx} (x) \right) dx \right]$$

$$= -x^2 \cos x + 2 \left[x \sin x - \int \sin x \, dx \right]$$

$$\begin{array}{l} \because x \rightarrow 1^{\text{st}} \text{ function} \\ \sin x \rightarrow 2^{\text{nd}} \text{ function} \end{array}$$

$$= -x^2 \cos x + 2x \sin x - 2(-\cos x) + c$$

$$= -x^2 \cos x + 2x \sin x + 2 \cos x + c \quad \text{Ans.}$$

(ix) $\int x^2 \tan^{-1} x \, dx$ (Lhr. Board 2006, 2008)

$$= \tan^{-1} x \int x^2 \, dx - \int \left[x^2 \, dx \cdot \frac{d}{dx} (\tan^{-1} x) \right] dx$$

$$\begin{array}{l} \because \tan^{-1} x \rightarrow 1^{\text{st}} \text{ function} \\ x^2 \rightarrow 2^{\text{nd}} \text{ function} \end{array}$$

$$= \tan^{-1} x \cdot \frac{x^3}{3} - \int \frac{x^3}{3} \cdot \frac{1}{1+x^2} \, dx$$

$$= \frac{x^3 \tan^{-1} x}{3} - \frac{1}{3} \int \frac{x^3}{1+x^2} \, dx$$

$$= \frac{x^3 \tan^{-1} x}{3} - \frac{1}{3} \int \left(x - \frac{x}{1+x^2} \right) dx$$

$$\begin{array}{l} \because \frac{x}{1+x^2} = \frac{x}{\sqrt{x^3}} \\ = \frac{x^3 \pm x}{-x} \end{array}$$

$$= \frac{x^3 \tan^{-1} x}{3} - \frac{1}{3} \int x \, dx + \frac{1}{3 \cdot 2} \int \frac{2x}{1+x^2} \, dx$$

$$\begin{array}{l} \because f(x) = 1+x^2 \\ f'(x) = 2x \end{array}$$



$$= \frac{x^3 \tan^{-1} x}{3} - \frac{1}{3} \frac{x^2}{2} + \frac{1}{6} \ln(1+x^2) + c$$

$$= \frac{x^3 \tan^{-1} x}{3} - \frac{x^2}{6} + \frac{1}{6} \ln(1+x^2) + c \quad \text{Ans.}$$

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(x) $\int x \tan^{-1} x \, dx$

$\therefore \tan^{-1} x \rightarrow 1^{\text{st}} \text{ function}$

$x \rightarrow \text{nd function}$

$$= \tan^{-1} x \int x \, dx - \int \left[\int x \, dx \cdot \frac{d}{dx} (\tan^{-1} x) \right] dx$$

$$= \tan^{-1} x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{1+x^2} dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int \frac{x^2}{1+x^2} dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2} \right) dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int dx + \frac{1}{2} \int \frac{dx}{1+x^2}$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} x + \frac{1}{2} \tan^{-1} x + c$$

$$= \left(\frac{1}{2} \tan^{-1} x \right) (x^2 + 1) - \frac{1}{2} x + c \quad \text{Ans.}$$

(xi) $\int x^3 \tan^{-1} x \, dx$ (Guj. Board 2007)

$\therefore \tan^{-1} x \rightarrow 1^{\text{st}} \text{ function}$

$x^3 \rightarrow \text{nd function}$

$$= \tan^{-1} x \int x^3 \, dx - \int \left[x^3 \, dx \cdot \frac{d}{dx} (\tan^{-1} x) \right] dx$$

$$= \tan^{-1} x \cdot \frac{x^4}{4} - \int \frac{x^4}{4} \cdot \frac{1}{1+x^2} dx$$

$$= \frac{x^4 \tan^{-1} x}{4} - \frac{1}{4} \int \frac{x^4}{1+x^2} dx$$

$$= \frac{x^4 \tan^{-1} x}{4} - \frac{1}{4} \int \left(x^2 - 1 + \frac{1}{1+x^2} \right) dx$$

$$= \frac{x^4 \tan^{-1} x}{4} - \frac{1}{4} \int x^2 \, dx + \frac{1}{4} \int dx - \frac{1}{4} \int \frac{dx}{1+x^2}$$

$$= \frac{x^4 \tan^{-1} x}{4} - \frac{1}{4} \cdot \frac{x^3}{3} + \frac{1}{4} x - \frac{1}{4} \tan^{-1} x + c$$

$$= \frac{1}{4} \left[(x^4 - 1) \tan^{-1} x - \frac{x^3}{3} + x \right] + c \quad \text{Ans.}$$

$$1 + x^2 \sqrt{\frac{x^2}{x^2 + 1}}$$

$$1 + x^2 \sqrt{\frac{x^2 - 1}{x^4 + x^2}}$$



(xii) $\int x^3 \cos x \, dx$

$\therefore x^3 \rightarrow 1^{\text{st}} \text{ function}$
 $\cos x \rightarrow 2^{\text{nd}} \text{ function}$

$$\begin{aligned}
 &= x^3 \int \cos x \, dx - \int \left[\cos x \, dx \cdot \frac{d}{dx}(x^3) \right] dx \\
 &= x^3 \sin x - \int \sin x \cdot 3x^2 \, dx \\
 &= x^3 \sin x - 3 \int x^2 \sin x \, dx \\
 &= x^3 \sin x - 3 \left[x^2 \int \sin x \, dx - \int \left(\sin x \, dx \cdot \frac{d}{dx}(x^2) \right) dx \right] \\
 &= x^3 \sin x - 3x^2 (-\cos x) + 3 \int -\cos x \cdot 2x \, dx \\
 &= x^3 \sin x + 3x^2 \cos x - 6 \int x \cos x \, dx \\
 &= x^3 \sin x + 3x^2 \cos x - 6 \left[x \int \cos x \, dx - \int \left(\cos x \, dx \cdot \frac{d}{dx}(x) \right) dx \right] \\
 &= x^3 \sin x + 3x^2 \cos x - 6x \sin x + 6 \int \sin x \, dx \\
 &= x^3 \sin x + 3x^2 \cos x - 6x \sin x + 6(-\cos x) + c \\
 &= (x^3 - 6x) \sin x + (3x^2 - 6) \cos x + c \quad \text{Ans.}
 \end{aligned}$$

(xiii) $\int \sin^{-1} x \, x \, dx$

$\therefore \sin^{-1} x \rightarrow 1^{\text{st}} \text{ function}$
 $x \rightarrow 2^{\text{nd}} \text{ function}$

$$\begin{aligned}
 &= \int \sin^{-1} x \cdot 1 \, dx \\
 &= \sin^{-1} x \int dx - \int \left[\int dx \cdot \frac{d}{dx}(\sin^{-1} x) \right] dx \\
 &= \sin^{-1} x \cdot x - \int x \cdot \frac{1}{\sqrt{1-x^2}} \, dx \\
 &= x \sin^{-1} x + \frac{1}{2} \int (1-x^2)^{-1/2} \cdot -2x \, dx \\
 &= x \sin^{-1} x + \frac{1}{2} \frac{(1-x^2)^{1/2}}{1/2} + c \\
 &= x \sin^{-1} x + \sqrt{1-x^2} + c \quad \text{Ans.}
 \end{aligned}$$



$$(xiv) \int x \sin^{-1} x dx$$

$$= \sin^{-1} x \int x dx - \int \left[x dx \cdot \frac{d}{dx} (\sin^{-1} x) \right] dx$$

$$\because \sin^{-1} x \rightarrow 1^{\text{st}} \text{ function} \\ x \rightarrow 2^{\text{nd}} \text{ function}$$

$$= \sin^{-1} x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{\sqrt{1-x^2}} dx$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \frac{1-x^2-1}{\sqrt{1-x^2}} dx$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \frac{1-x^2}{\sqrt{1-x^2}} dx - \frac{1}{2} \int \frac{dx}{\sqrt{1-x^2}}$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \sqrt{1-x^2} dx - \frac{1}{2} \sin^{-1} x$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} I - \frac{1}{2} \sin^{-1} x \quad \text{———— (1)}$$

Taking

$$I = \int \sqrt{1-x^2} dx$$

Put

$$x = \sin \theta \Rightarrow \theta = \sin^{-1} x$$

$$dx = \cos \theta d\theta$$

$$I = \int \sqrt{1-\sin^2 \theta} \cdot \cos \theta d\theta$$

$$I = \int \sqrt{\cos^2 \theta} \cdot \cos \theta d\theta$$

$$I = \int \cos \theta \cdot \cos \theta d\theta$$

$$I = \int \cos^2 \theta d\theta$$

$$I = \int \frac{1 + \cos 2\theta}{2} d\theta \quad \left(\begin{array}{l} \because \cos^2 \theta = 2\cos^2 \theta - 1 \\ 2\cos^2 \theta = 1 + \cos 2\theta \\ \cos^2 \theta = \frac{1 + \cos 2\theta}{2} \end{array} \right)$$

$$I = \frac{1}{2} \int (1 + \cos 2\theta) d\theta$$



$$I = \frac{1}{2} \int d\theta + \frac{1}{2.2} \int \cos 2\theta \cdot 2d\theta$$

$$I = \frac{1}{2} \theta + \frac{1}{4} \sin 2\theta + c$$

$$I = \frac{1}{2} \sin^{-1} x + \frac{1}{4} \cdot 2 \sin \theta \cos \theta + c$$

$$I = \frac{1}{2} \sin^{-1} x + \frac{1}{2} x \sqrt{\cos^2 \theta} + c$$

$$I = \frac{1}{2} \sin^{-1} x + \frac{1}{2} x \sqrt{1 - \sin^2 \theta} + c$$

$$I = \frac{1}{2} \sin^{-1} x + \frac{1}{2} x \sqrt{1 - x^2} + c$$

∴ From equation (1)

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \left[\frac{1}{2} \sin^{-1} x + \frac{1}{2} x \sqrt{1 - x^2} \right] - \frac{1}{2} \sin^{-1} x + c$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{4} \sin^{-1} x + \frac{x \sqrt{1 - x^2}}{4} - \frac{1}{2} \sin^{-1} x + c$$

$$= \frac{x^2 \sin^{-1} x}{2} - \frac{1}{4} \sin^{-1} x + \frac{x \sqrt{1 - x^2}}{4} + c \quad \text{Ans.}$$

(xv) $\int e^x \sin x \cos x \, dx$

$$= \frac{1}{2} \int e^x 2 \sin x \cos x \, dx$$

$$= \frac{1}{2} \int e^x \sin 2x \, dx$$

$$= \frac{1}{2} I \quad (1)$$

Where

$$I = \int e^x \sin 2x \, dx$$

$$= e^x \cdot \frac{-\cos 2x}{2} - \int \frac{-\cos 2x}{2} \cdot e^x \, dx$$

$$= -\frac{e^x \cos 2x}{2} + \frac{1}{2} \int e^x \cos 2x \, dx$$

$\therefore e^x \rightarrow 1^{\text{st}} \text{ function}$ $\sin 2x \rightarrow 2^{\text{nd}} \text{ function}$



$$= -\frac{e^x \cos 2x}{2} + \frac{1}{2} \left[e^x \frac{\sin 2x}{2} - \int \frac{\sin 2x}{2} \cdot e^x dx \right]$$

$$I = -\frac{e^x \cos 2x}{2} + \frac{e^x \sin 2x}{4} - \frac{1}{4} \int e^x \sin 2x dx$$

$$I = -\frac{e^x \cos 2x}{2} + \frac{e^x \sin 2x}{4} - \frac{1}{4} I + c$$

$$I + \frac{I}{4} = -\frac{e^x \cos 2x}{2} + \frac{e^x \sin 2x}{4} + c$$

$$\frac{5I}{4} = -\frac{e^x \cos 2x}{2} + \frac{e^x \sin 2x}{4} + c$$

$$I = -\frac{4e^x \cos 2x}{10} + \frac{4e^x \sin 2x}{20} + c$$

$$I = -\frac{2e^x \cos 2x}{5} + \frac{e^x \sin 2x}{5} + c$$

$\therefore e^x \rightarrow 1^{\text{st}} \text{ function}$
 $\cos 2x \rightarrow 2^{\text{nd}} \text{ function}$

From equation (1)

$$= \frac{1}{2} \left(-\frac{2e^x \cos 2x}{5} + \frac{e^x \sin 2x}{5} \right) + c$$

$$= \frac{1}{5} e^x \left(-\cos 2x + \frac{\sin 2x}{2} \right) + c$$

$$= \frac{1}{5} e^x \left[-(1 - 2\sin^2 x) + \frac{\sin 2x}{2} \right] + c$$

$$= \frac{1}{5} e^x (2\sin^2 x - 1 + \frac{1}{2} \sin 2x) + c \quad \text{Ans.}$$

(xvi) $\int x \sin x \cos x dx$

$$= \frac{1}{2} \int x \cdot 2 \sin x \cos x dx$$

$$= \frac{1}{2} \int x \sin 2x dx$$

$$= \frac{1}{2} \left[x \cdot \frac{-\cos 2x}{2} - \int \frac{-\cos 2x}{2} \cdot 1 dx \right]$$

$$= \frac{-x \cos 2x}{4} + \frac{1}{4} \int \cos 2x dx$$

$$= \frac{-x \cos 2x}{4} + \frac{1}{4} \cdot \frac{\sin 2x}{2} + c$$

$$= \frac{1}{4} \left(-x \cos 2x + \frac{2 \sin x \cos x}{2} \right) + c$$

$\therefore x \rightarrow 1^{\text{st}} \text{ function}$
 $\sin 2x \rightarrow 2^{\text{nd}} \text{ function}$



$$= \frac{1}{4} (\sin x \cos x - x \cos 2x) + c \quad \text{Ans.}$$

$$\text{(xvii)} \quad \int x \cos^2 x \, dx$$

$$= \int x \left(\frac{1 + \cos 2x}{2} \right) dx$$

$$= \frac{1}{2} \int (x + x \cos 2x) dx$$

$$= \frac{1}{2} \int x dx + \frac{1}{2} \int x \cos 2x dx$$

$$= \frac{1}{2} \cdot \frac{x^2}{2} + \frac{1}{2} \left[x \cdot \frac{\sin 2x}{2} - \int \frac{\sin 2x}{2} \cdot 1 dx \right]$$

$$= \frac{x^2}{4} + \frac{x \sin 2x}{4} - \frac{1}{4} \int \sin 2x dx$$

$$= \frac{x^2}{4} + \frac{x \sin 2x}{4} - \frac{1}{4} \cdot \left(\frac{-\cos 2x}{2} \right) + c$$

$$= \frac{1}{4} (x^2 + x \sin 2x + \frac{\cos 2x}{2}) + c \quad \text{Ans.}$$

$\therefore x \rightarrow 1^{\text{st}} \text{ function}$
 $\cos 2x \rightarrow 2^{\text{nd}} \text{ function}$

$$\text{(xviii)} \quad \int x \sin^2 x dx$$

$$= \int x \left(\frac{1 - \cos 2x}{2} \right) dx$$

$$\therefore \cos 2x = 1 - 2\sin^2 x$$

$$2\sin^2 x = 1 - \cos 2x$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$= \frac{1}{2} \int (x - x \cos 2x) dx$$

$$= \frac{1}{2} \int x dx - \frac{1}{2} \int x \cos 2x dx$$

$$= \frac{1}{2} \cdot \frac{x^2}{2} - \frac{1}{2} \left[x \cdot \frac{\sin 2x}{2} - \int \frac{\sin 2x}{2} \cdot 1 dx \right]$$

$$= \frac{x^2}{4} - \frac{x \sin 2x}{4} + \frac{1}{4} \int \sin 2x dx$$

$$= \frac{x^2}{4} - \frac{x \sin 2x}{4} + \frac{1}{4} \left[\frac{-\cos 2x}{2} \right] + c$$

$\therefore x \rightarrow 1^{\text{st}} \text{ function}$
 $\cos 2x \rightarrow 2^{\text{nd}} \text{ function}$



$$= \frac{x^2}{4} - \frac{x \sin 2x}{4} - \frac{\cos 2x}{8} + c$$

$$= \frac{1}{4} (x^2 - x \sin 2x - \frac{1}{2} \cos 2x) + c \quad \text{Ans.}$$

(xix) $\int (\ln x)^2 dx$

$$= \int (\ln x)^2 \cdot 1 dx$$

$\because (\ln x)^2 \rightarrow 1^{\text{st}} \text{ function}$ $1 \rightarrow 2^{\text{nd}} \text{ function}$

$$= (\ln x)^2 \int dx - \int \left[\int dx \cdot \frac{d}{dx} (\ln x)^2 \right] dx$$

$$= (\ln x)^2 \cdot x - \int x \cdot 2 (\ln x) \cdot \frac{1}{x} dx$$

$$= x (\ln x)^2 - 2 \int \ln x \cdot 1 dx$$

$$= x (\ln x)^2 - 2 \left[\ln x dx - \int \left(\int dx \cdot \frac{d}{dx} (\ln x) \right) dx \right]$$

$$= x (\ln x)^2 - 2 \ln x \cdot x + 2 \int x \cdot \frac{1}{x} dx$$

$$= x (\ln x)^2 - 2x \ln x + 2 \int dx$$

$$= x (\ln x)^2 - 2x \ln x + 2x + c$$

$$= x \ln x (\ln x - 2) + 2x + c \quad \text{Ans.}$$

(xx) $\int \ln (\tan x) \sec^2 x dx$

$$= \ln (\tan x) \int \sec^2 x dx - \int \left[\int \sec^2 x dx \cdot \frac{d}{dx} \ln (\tan x) \right] dx$$

$$= \ln (\tan x) \cdot \tan x - \int \tan x \cdot \frac{1}{\tan x} \cdot \sec^2 x dx$$

$$= \tan x \ln (\tan x) - \int \sec^2 x dx$$

$$= \tan x \ln (\tan x) - \tan x + c \quad \text{Ans.}$$

Alternate Method

Solution:

$$t = \tan x$$

$$dt = \sec^2 x dx$$



$$\begin{aligned}
 &= \int \ln t \cdot 1 \, dt \\
 &= \ln t \cdot t - \int t \cdot \frac{1}{t} \, dt \\
 &= t \cdot \ln t - \int dt \\
 &= t \cdot \ln t - t + c \\
 &= \tan x \cdot \ln (\tan x) - \tan x + c
 \end{aligned}$$

(xxi) $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} \, dx$

Put

$$\sin^{-1} x = t \quad \Rightarrow \quad x = \sin t$$

$$\frac{1}{\sqrt{1-x^2}} \, dx = dt$$

$$dx = \sqrt{1-x^2} \, dt$$

$$= \int \frac{\sin t \times t}{\sqrt{1-x^2}} \times \sqrt{1-x^2} \, dt$$

$$= \int t \sin t \, dt$$

$$= t \int \sin t \, dt - \int \left[\int \sin t \, dt \cdot \frac{d}{dt} (t) \right] dt$$

$$= t \cdot -\cos t - \int -\cos t \cdot 1 \, dt$$

$$= -t \cos t + \sin t + c$$

$$= -\sin^{-1} x \sqrt{\cos^2 t} + x + c$$

$$= x - \sin^{-1} x \sqrt{1 - \sin^2 t} + c$$

$$= x - \sqrt{1-x^2} \sin^{-1} x + c \quad \text{Ans.}$$

$$\begin{array}{ll}
 \because t & \rightarrow 1^{\text{st}} \text{ function} \\
 \sin t & \rightarrow 2^{\text{nd}} \text{ function}
 \end{array}$$

$$\because t = \sin^{-1} x$$

Q.2 Evaluate the following integral.

(i) $\int \tan^4 x \, dx$ (L.B 2009) (ii) $\int \sec^4 x \, dx$ (iii) $\int e^x \sin 2x \cdot \cos x \, dx$

(iv) $\int \tan^3 x \sec x \, dx$ (v) $\int x^3 e^{5x} \, dx$ (vi) $\int e^{-x} \sin 2x \, dx$



(vii) $\int e^{2x} \cos 3x dx$

(viii) $\int \operatorname{cosec}^3 x dx$

Solution:

(i) $\int \tan^4 x dx$

$$\begin{aligned}
 &= \int \tan^2 x \cdot \tan^2 x dx \\
 &= \int \tan^2 x (\sec^2 x - 1) dx \\
 &= \int (\tan^2 x \sec^2 x - \tan^2 x) dx \\
 &= \int \tan^2 x \sec^2 x dx - \int (\sec^2 x - 1) dx \\
 &= \frac{\tan^3 x}{3} - \int \sec^2 x dx + \int dx \\
 &= \frac{\tan^3 x}{3} - \tan x + x + c \quad \text{Ans.}
 \end{aligned}$$

(ii) $\int \sec^4 x dx$

$$\begin{aligned}
 &= \int \sec^2 x \cdot \sec^2 x dx \\
 &= \int \sec^2 x (1 + \tan^2 x) dx \\
 &= \int (\sec^2 x + \sec^2 x \tan^2 x) dx \\
 &= \int \sec^2 x dx + \int \tan^2 x \sec^2 x dx \\
 &= \tan x + \frac{\tan^3 x}{3} + c \quad \text{Ans.}
 \end{aligned}$$

(iii) $\int e^x \sin 2x \cos x dx$

$$\begin{aligned}
 &= \frac{1}{2} \int e^x 2 \sin 2x \cos x dx \\
 &= \frac{1}{2} \int e^x [\sin(2x + x) + \sin(2x - x)] dx \\
 &= \frac{1}{2} \int (e^x \sin 3x + e^x \sin x) dx \\
 &= \frac{1}{2} \int e^x \sin 3x dx + \frac{1}{2} \int e^x \sin x dx \\
 &= \frac{1}{2} I_1 + \frac{1}{2} I_2 \quad \text{—————} \quad (2)
 \end{aligned}$$

$\therefore 2 \sin \alpha \cos \beta = \sin(\alpha + \beta) + \sin(\alpha - \beta)$



$$I_1 = \int e^x \sin 3x dx$$

$$I_1 = e^x \int \sin 3x dx - \int \left[\int \sin 3x dx \cdot \frac{d}{dx} (e^x) \right] dx$$

$$I_1 = e^x \left(\frac{-\cos 3x}{3} \right) - \int \frac{-\cos 3x}{3} \cdot e^x dx$$

$$I_1 = -\frac{e^x \cos 3x}{3} + \frac{1}{3} \int e^x \cos 3x dx$$

$$I_1 = \frac{e^x \cos 3x}{3} + \frac{1}{3} \left[e^x \int \cos 3x dx - \int \left(\int \cos 3x dx \cdot \frac{d}{dx} (e^x) \right) dx \right]$$

$$I_1 = -\frac{e^x \cos 3x}{3} + \frac{1}{3} e^x \cdot \frac{\sin 3x}{3} - \frac{1}{3} \int \frac{\sin 3x}{3} \cdot e^x dx$$

$$I_1 = -\frac{e^x \cos 3x}{3} + \frac{e^x \sin 3x}{9} - \frac{1}{9} \int e^x \sin 3x dx$$

$$I_1 = -\frac{e^x \cos 3x}{3} + \frac{e^x \sin 3x}{9} - \frac{1}{9} I_1 + c_1$$

$$I_1 + \frac{1}{9} I_1 = -\frac{e^x \cos 3x}{3} + \frac{e^x \sin 3x}{9} + c_1$$

$$\frac{10}{9} I_1 = -\frac{e^x \cos 3x}{3} + \frac{e^x \sin 3x}{9} + c_1$$

$$I_1 = -\frac{9e^x \cos 3x}{30} + \frac{9e^x \sin 3x}{90} + c_1$$

$$I_1 = -\frac{3e^x \cos 3x}{10} + \frac{e^x \sin 3x}{10} + c_1$$

$$I_2 = \int e^x \sin x dx$$

$$I_2 = e^x \int \sin x dx - \int \left[\int \sin x dx \cdot \frac{d}{dx} (e^x) \right] dx$$

$$I_2 = e^x (-\cos x) - \int -\cos x \cdot e^x dx$$

$$I_2 = -e^x \cos x + \int e^x \cos x dx$$

$$I_2 = -e^x \cos x + e^x \int \cos x dx - \int \left[\int \cos x dx \cdot \frac{d}{dx} (e^x) \right] dx$$

$$I_2 = -e^x \cos x + e^x \sin x - \int \sin x e^x dx$$

$$I_2 = -e^x \cos x + e^x \sin x - I_2 + c_2$$



$$2I_2 = -e^x \cos x + e^x \sin x + c_2$$

$$I_2 = \frac{-e^x \cos x}{2} + \frac{e^x \sin x}{2} + c_2$$

∴ From eq. (1)

$$= \frac{1}{2} \left[-\frac{3e^x \cos 3x}{10} + \frac{e^x \sin 3x}{10} + c_1 \right] + \frac{1}{2} \left[\frac{-e^x \cos x}{2} + \frac{e^x \sin x}{2} + c_2 \right]$$

$$= \frac{-3e^x \cos 3x}{20} + \frac{e^x \sin 3x}{20} + c_1 + \frac{-e^x \cos x}{4} + \frac{e^x \sin x}{4} + c_2$$

$$= \frac{1}{4} e^x \left[\frac{1}{5} \sin 3x - \frac{3}{5} \cos 3x + \sin x - \cos x \right] + c \quad \text{where } c = c_1 + c_2$$

(iv) $\int \tan^3 x \sec x dx$

$$= \int \tan^2 x \cdot \sec x \tan x dx$$

$$= \int (\sec^2 x - 1) \sec x \tan x dx$$

$$= \int \sec^2 x \cdot \sec x \tan x dx - \int \sec x \tan x dx$$

$$= \frac{\sec^3 x}{3} - \sec x + c \quad \text{Ans.}$$

(v) $\int x^3 e^{5x} dx$

$$= x^3 \int e^{5x} dx - \int \left[\int e^{5x} dx \cdot \frac{d}{dx} (x^3) \right] dx$$

$$= x^3 \cdot \frac{e^{5x}}{5} - \int \frac{e^{5x}}{5} \cdot 3x^2 dx$$

$$= \frac{x^3 e^{5x}}{5} - \frac{3}{5} \int x^2 e^{5x} dx$$

$$= \frac{x^3 e^{5x}}{5} - \frac{3}{5} \left[x^2 \int e^{5x} dx - \int \left(\int e^{5x} dx \cdot \frac{d}{dx} (x^2) \right) dx \right]$$

$$= \frac{x^3 e^{5x}}{5} - \frac{3}{5} x^2 \cdot \frac{e^{5x}}{5} + \frac{3}{5} \int \frac{e^{5x}}{5} \cdot 2x dx$$

$$= \frac{x^3 e^{5x}}{5} - \frac{3x^2 e^{5x}}{25} + \frac{6}{25} \int x e^{5x} dx$$

$$= \frac{x^3 e^{5x}}{5} - \frac{3x^2 e^{5x}}{25} + \frac{6}{25} \left[x \int e^{5x} dx - \int \left(\int e^{5x} dx \cdot \frac{d}{dx} (x) \right) dx \right]$$



$$\begin{aligned}
&= \frac{x^3 e^{5x}}{5} - \frac{3x^2 e^{5x}}{25} + \frac{6}{25} x \cdot \frac{e^{5x}}{5} - \frac{6}{25} \int \frac{e^{5x}}{5} \cdot 1 dx \\
&= \frac{x^3 e^{5x}}{5} - \frac{3x^2 e^{5x}}{25} + \frac{6x e^{5x}}{125} - \frac{6}{125} \int e^{5x} dx \\
&= \frac{x^3 e^{5x}}{5} - \frac{3x^2 e^{5x}}{25} + \frac{6x e^{5x}}{125} - \frac{6}{125} \cdot \frac{e^{5x}}{5} + c \\
&= \frac{1}{5} e^{5x} \left(x^3 - \frac{3}{5} x^2 + \frac{6x}{25} - \frac{6}{125} \right) + c \quad \text{Ans.}
\end{aligned}$$

(vi) $\int e^{-x} \sin 2x \, dx$

Let

$$\begin{aligned}
I &= \int e^{-x} \sin 2x \, dx \\
I &= e^{-x} \int \sin 2x \, dx - \int \left[\int \sin 2x \, dx \cdot \frac{d}{dx} (e^{-x}) \right] dx \\
I &= e^{-x} \frac{-\cos 2x}{2} - \int \frac{-\cos 2x}{2} \cdot -e^{-x} dx \\
I &= -\frac{e^{-x} \cos 2x}{2} - \frac{1}{2} \int e^{-x} \cos 2x \, dx \\
I &= -\frac{e^{-x} \cos 2x}{2} - \frac{1}{2} \left[e^{-x} \int \cos 2x \, dx - \int \left(\int \cos 2x \, dx \cdot \frac{d}{dx} (e^{-x}) \right) dx \right] \\
I &= -\frac{e^{-x} \cos 2x}{2} - \frac{1}{2} e^{-x} \frac{\sin 2x}{2} + \frac{1}{2} \int \frac{\sin 2x}{2} (-e^{-x}) dx \\
I &= -\frac{e^{-x} \cos 2x}{2} - \frac{e^{-x} \sin 2x}{4} - \frac{1}{4} \int e^{-x} \sin 2x \, dx \\
I &= -\frac{e^{-x} \cos 2x}{2} - \frac{e^{-x} \sin 2x}{4} - \frac{1}{4} I + c \\
I + \frac{1}{4} I &= -\frac{e^{-x} \cos 2x}{2} - \frac{e^{-x} \sin 2x}{4} + c \\
\frac{5}{4} I &= -\frac{e^{-x} \cos 2x}{2} - \frac{e^{-x} \sin 2x}{4} + c \\
I &= \frac{-4e^{-x} \cos 2x}{10} - \frac{4e^{-x} \sin 2x}{20} + c \\
I &= -\frac{2e^{-x} \cos 2x}{5} - \frac{e^{-x} \sin 2x}{5} + c \\
I &= -\frac{2}{5} e^{-x} \left(\cos 2x + \frac{\sin 2x}{2} \right) + c \quad \text{Ans.}
\end{aligned}$$



(vii) $\int e^{2x} \cos 3x dx$

Let

$$I = \int e^{2x} \cos 3x dx$$

$$I = e^{2x} \int \cos 3x dx - \int \left[\int \cos 3x dx \cdot \frac{d}{dx} (e^{2x}) \right] dx$$

$$I = e^{2x} \cdot \frac{\sin 3x}{3} - \int \frac{\sin 3x}{3} \cdot e^{2x} \cdot 2 dx$$

$$I = \frac{e^{2x} \sin 3x}{3} - \frac{2}{3} \int e^{2x} \sin 3x dx$$

$$I = \frac{e^{2x} \sin 3x}{3} - \frac{2}{3} \left[e^{2x} \int \sin 3x dx - \int \left(\int \sin 3x dx \cdot \frac{d}{dx} (e^{2x}) \right) dx \right]$$

$$I = \frac{e^{2x} \sin 3x}{3} - \frac{2}{3} e^{2x} \left(\frac{-\cos 3x}{3} \right) + \frac{2}{3} \int \frac{-\cos 3x}{3} \cdot 2e^{2x} dx$$

$$I = \frac{e^{2x} \sin 3x}{3} + \frac{2e^{2x} \cos 3x}{9} - \frac{4}{9} \int e^{2x} \cos 3x dx$$

$$I = \frac{e^{2x} \sin 3x}{3} + \frac{2e^{2x} \cos 3x}{9} - \frac{4}{9} I + c$$

$$I + \frac{4}{9} I = \frac{e^{2x} \sin 3x}{3} + \frac{2e^{2x} \cos 3x}{9} + c$$

$$\frac{13}{9} I = \frac{e^{2x} \sin 3x}{3} + \frac{2e^{2x} \cos 3x}{9} + c$$

$$I = \frac{9e^{2x} \sin 3x}{3 \times 13} + \frac{18e^{2x} \cos 3x}{13 \times 9} + c$$

$$I = \frac{3e^{2x} \sin 3x}{13} + \frac{2e^{2x} \cos 3x}{13} + c$$

$$I = \frac{3}{13} e^{2x} \left(\sin 3x + \frac{2}{3} \cos 3x \right) + c \quad \text{Ans.}$$

(viii) $\int \operatorname{cosec}^3 x dx$

Let

$$I = \int \operatorname{cosec}^3 x dx$$

$$I = \int \operatorname{cosec}^2 x \cdot \operatorname{cosec} x dx$$



$$\begin{aligned}
 I &= \operatorname{cosec} x \int \operatorname{cosec}^2 x dx - \int \left[\int \operatorname{cosec}^2 x dx \cdot \frac{d}{dx} (\operatorname{cosec} x) \right] dx \\
 I &= \operatorname{cosec} x \cdot -\cot x - \int -\cot x \cdot -\operatorname{cosec} x \cot x dx \\
 I &= -\operatorname{cosec} x \cot x - \int \operatorname{cosec} x \cot^2 x dx \\
 I &= -\operatorname{cosec} x \cot x - \int \operatorname{cosec} x (\operatorname{cosec}^2 x - 1) dx \\
 I &= -\operatorname{cosec} x \cot x - \int \operatorname{cosec}^3 x dx + \int \operatorname{cosec} x dx \\
 I &= -\operatorname{cosec} x \cot x - I + \ln |\operatorname{cosec} x - \cot x| + c \\
 2I &= -\operatorname{cosec} x \cot x + \ln |\operatorname{cosec} x - \cot x| + c \\
 I &= \frac{-1}{2} [\operatorname{cosec} x \cot x - \ln |\operatorname{cosec} x - \cot x|] + c \quad \text{Ans.}
 \end{aligned}$$

Q.3 Show that $\int e^{ax} \sin bx dx = \frac{1}{\sqrt{a^2 + b^2}} e^{ax} \sin (bx - \tan^{-1} \frac{b}{a}) + c$ (Guj. Board 2005)

Solution:

Taking

$$\int e^{ax} \sin bx dx$$

Let

$$I = \int e^{ax} \sin bx dx$$

$$\begin{array}{l}
 \therefore e^{ax} \rightarrow 1^{\text{st}} \text{ function} \\
 \sin bx \rightarrow 2^{\text{nd}} \text{ function}
 \end{array}$$

$$I = e^{ax} \int \sin bx dx - \int \left[\int \sin bx dx \cdot \frac{d}{dx} (e^{ax}) \right] dx$$

$$I = e^{ax} \left(\frac{-\cos bx}{b} \right) - \int \frac{-\cos bx}{b} \cdot e^{ax} \cdot a dx$$

$$I = \frac{-e^{ax} \cos bx}{b} + \frac{a}{b} \int e^{ax} \cos bx dx$$

$$I = \frac{-e^{ax} \cos bx}{b} + \frac{a}{b} \left[e^{ax} \int \cos bx dx - \int \left(\int \cos bx dx \cdot \frac{d}{dx} (e^{ax}) \right) dx \right]$$

$$I = \frac{-e^{ax} \cos bx}{b} + \frac{a}{b} e^{ax} \cdot \frac{\sin bx}{b} - \frac{a}{b} \int \frac{\sin bx}{b} \cdot a e^{ax} dx$$

$$I = \frac{-e^{ax} \cos bx}{b} + \frac{ae^{ax} \sin bx}{b^2} - \frac{a^2}{b^2} \int e^{ax} \sin bx dx$$

$$I = \frac{-e^{ax} \cos bx}{b} + \frac{ae^{ax} \sin bx}{b^2} - \frac{a^2}{b^2} I + c$$

$$\begin{array}{l}
 \therefore e^{ax} \rightarrow 1^{\text{st}} \text{ function} \\
 \cos bx \rightarrow 2^{\text{nd}} \text{ function}
 \end{array}$$



$$\begin{aligned}
 I + \frac{a^2}{b^2} I &= -\frac{e^{ax} \cos bx}{b} + \frac{ae^{ax} \sin bx}{b^2} + c \\
 \left(\frac{a^2 + b^2}{b^2}\right) I &= -\frac{e^{ax} \cos bx}{b} + \frac{ae^{ax} \sin bx}{b^2} + c \\
 I &= \frac{-b^2 e^{ax} \cos bx}{b(a^2 + b^2)} + \frac{ab^2 e^{ax} \sin bx}{b^2(a^2 + b^2)} + c \\
 I &= \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + c
 \end{aligned}$$

Put

$$a = r \cos \theta \quad \text{————— (1)}$$

$$b = r \sin \theta \quad \text{————— (2)}$$

Squaring and adding equation (1) and equation (2)

$$a^2 + b^2 = r^2 \cos^2 \theta + r^2 \sin^2 \theta$$

$$a^2 + b^2 = r^2 (\cos^2 \theta + \sin^2 \theta)$$

$$r^2 = a^2 + b^2$$

$$r = \sqrt{a^2 + b^2}$$

Dividing equation (2) by equation (1)

$$\frac{b}{a} = \frac{r \sin \theta}{r \cos \theta}$$

$$\tan \theta = \frac{b}{a}$$

$$\theta = \tan^{-1} \frac{b}{a}$$

$$I = \frac{e^{ax}}{a^2 + b^2} [\sin bx \cdot r \cos \theta - r \sin \theta \cos bx] + c$$

$$I = \frac{re^{ax}}{a^2 + b^2} [\sin bx \cos \theta - \cos bx \sin \theta] + c$$

$$I = \frac{\sqrt{a^2 + b^2} e^{ax}}{a^2 + b^2} [\sin (bx - \theta)] + c$$

$$I = \frac{e^{ax}}{\sqrt{a^2 + b^2}} \left[\sin \left(bx - \tan^{-1} \frac{b}{a} \right) \right] + c \quad \text{Hence proved.}$$

Q.4 Evaluate the following indefinite integrals.

$$(i) \int \sqrt{a^2 - x^2} \, dx \quad (ii) \int \sqrt{x^2 - a^2} \, dx \quad (iii) \int \sqrt{4 - 5x^2} \, dx$$



$$(iv) \int \sqrt{3-4x^2} \, dx \quad (v) \int \sqrt{x^2+4} \, dx \quad (vi) \int x^2 e^{ax} \, dx$$

Solution:

(i) $\int \sqrt{a^2 - x^2} \, dx$ (Lhr. Board 2007, 2009) (Guj. Board 2008)

Put $x = a \sin \theta \Rightarrow \sin \theta = \frac{x}{a}$

$dx = a \cos \theta \, d\theta \Rightarrow \theta = \sin^{-1} \left(\frac{x}{a} \right)$

$$= \int \sqrt{a^2 - a^2 \sin^2 \theta} \cdot a \cos \theta \, d\theta$$

$$= \int \sqrt{a^2 (1 - \sin^2 \theta)} \cdot a \cos \theta \, d\theta$$

$$= \int a \sqrt{\cos^2 \theta} \cdot a \cos \theta \, d\theta$$

$$= a^2 \int \cos \theta \cdot \cos \theta \, d\theta$$

$$= a^2 \int \cos^2 \theta \, d\theta$$

$$= a^2 \int \frac{1 + \cos 2\theta}{2} \, d\theta$$

$$= \frac{a^2}{2} \int (1 + \cos 2\theta) \, d\theta$$

$$= \frac{a^2}{2} \int d\theta + \frac{a^2}{2} \int \cos 2\theta \, d\theta$$

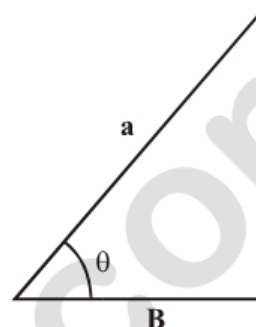
$$= \frac{a^2}{2} \theta + \frac{a^2}{2} \cdot \frac{\sin 2\theta}{2} + c$$

$$= \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + \frac{a^2}{4} 2 \sin \theta \cos \theta + c$$

$$= \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + \frac{a^2}{2} \cdot \frac{x}{a} \cos \theta + c$$

$$= \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + \frac{ax}{2} \cdot \frac{\sqrt{a^2 - x^2}}{a} + c$$

$$= \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + \frac{x \sqrt{a^2 - x^2}}{2} + c \quad \text{Ans.}$$



$$\begin{aligned} a^2 &= B^2 + x^2 \\ B^2 &= a^2 - x^2 \\ B &= \sqrt{a^2 - x^2} \end{aligned}$$

Alternative Solution

(i) $\int \sqrt{a^2 - x^2} \cdot 1 \cdot dx$



$$\therefore \sqrt{a^2 - x^2} \rightarrow 1^{\text{st}} \text{ function}$$

$$1 \rightarrow 2^{\text{nd}} \text{ function}$$

Now

$$= \sqrt{a^2 - x^2} \int 1 dx - \int \left[\int 1 dx \cdot \frac{d}{dx} \sqrt{a^2 - x^2} \right] dx$$

$$= \sqrt{a^2 - x^2} \cdot x - \int x \cdot \frac{-1(2x)}{2\sqrt{a^2 - x^2}} dx$$

$$= \sqrt{a^2 - x^2} \cdot x - \int \frac{-x^2}{\sqrt{a^2 - x^2}} dx$$

$$= x \cdot \sqrt{a^2 - x^2} - \int \frac{-a^2 + a^2 - x^2}{\sqrt{a^2 - x^2}} dx$$

$$= x \sqrt{a^2 - x^2} + a^2 \int \frac{1}{\sqrt{a^2 - x^2}} dx - \int \sqrt{a^2 - x^2} dx$$

$$= x \sqrt{a^2 - x^2} + a^2 \cdot \sin^{-1} \left(\frac{x}{a} \right) - I$$

$$2I = x \sqrt{a^2 - x^2} + a^2 \cdot \sin^{-1} \left(\frac{x}{a} \right) + c$$

Dividing both sides by 2

$$I = \frac{x \sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + c$$

$$(ii) \quad \sqrt{x^2 - a^2} \, dx$$

$$\text{Put } x = a \sec \theta \Rightarrow \sec \theta = \frac{x}{a}$$

$$dx = a \sec \theta \tan \theta \, d\theta$$

$$= \int \sqrt{a^2 \sec^2 \theta - a^2} \cdot a \sec \theta \tan \theta \, d\theta$$

$$= \int \sqrt{a^2 (\sec^2 \theta - 1)} \cdot a \sec \theta \tan \theta \, d\theta$$

$$= \int a \sqrt{\tan^2 \theta} \cdot a \sec \theta \tan \theta \, d\theta$$

$$= a^2 \int \tan \theta \cdot \sec \theta \tan \theta \, d\theta$$

$$= a^2 \int \tan^2 \theta \sec \theta \, d\theta$$

$$= a^2 \int (\sec^2 \theta - 1) \sec \theta \, d\theta$$



$$\begin{aligned}
 &= a^2 \int \sec^3 \theta d\theta - a^2 \int \sec \theta d\theta \\
 &= a^2 I - a^2 \ln |\sec \theta + \tan \theta| + c_1 \quad \text{—————} \quad (1)
 \end{aligned}$$

Where

$$\begin{aligned}
 I &= \int \sec^3 \theta d\theta \\
 I &= \int \sec \theta \cdot \sec^2 \theta d\theta \\
 I &= \sec \theta \int \sec^2 \theta d\theta - \int \left[\int \sec^2 \theta d\theta \cdot \frac{d}{d\theta} (\sec \theta) \right] d\theta \\
 I &= \sec \theta \tan \theta - \int \tan \theta \cdot \sec \theta \tan \theta d\theta \\
 I &= \sec \theta \tan \theta - \int \sec \theta \tan^2 \theta d\theta \\
 I &= \sec \theta \tan \theta - \int \sec \theta (\sec^2 \theta - 1) d\theta \\
 I &= \sec \theta \tan \theta - \sec^3 \theta d\theta + \int \sec \theta d\theta \\
 I &= \sec \theta \tan \theta - I + \ln |\sec \theta + \tan \theta| + c_2 \\
 2I &= \sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| + c_2 \\
 I &= \frac{\sec \theta \tan \theta}{2} + \frac{1}{2} \ln |\sec \theta + \tan \theta| + c_2
 \end{aligned}$$

∴ From equation (1)

$$= \frac{a^2 \sec \theta \tan \theta}{2} + \frac{a^2}{2} \ln |\sec \theta + \tan \theta| - a^2 \ln |\sec \theta + \tan \theta| + c_1 + c_2$$

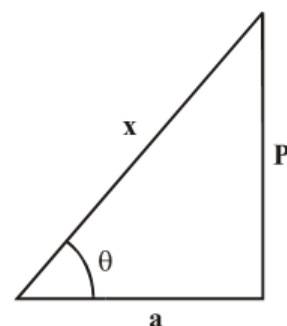
$$= \frac{a^2 \sec \theta \tan \theta}{2} + \left(\frac{a^2}{2} - a^2 \right) \ln |\sec \theta + \tan \theta| + c_1 + c_2$$

$$= \frac{a^2}{2} \cdot \frac{x}{a} \cdot \frac{\sqrt{x^2 - a^2}}{a} - \frac{a^2}{2} \ln \left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| + c_1 + c_2$$

$$= \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \ln \left| \frac{x + \sqrt{x^2 - a^2}}{a} \right| + c_1 + c_2$$

$$= \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \ln |x + \sqrt{x^2 - a^2}| + \frac{a^2}{2} \ln a + c_1 + c_2$$

$$= \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \ln |x + \sqrt{x^2 - a^2}| + c \text{ Ans. where } c = \frac{a^2}{2} \ln a + c_1 + c_2$$



$$\begin{aligned}
 x^2 &= a^2 + P^2 \\
 P^2 &= x^2 - a^2 \\
 P &= \sqrt{x^2 - a^2}
 \end{aligned}$$



Alternative Method

$$(ii) \quad \int \sqrt{x^2 - a^2} \, dx$$

$\therefore \sqrt{x^2 - a^2} \rightarrow 1^{\text{st}} \text{ function}$
 $1 \rightarrow 2^{\text{nd}} \text{ function}$

Let

$$\begin{aligned} I &= \int \sqrt{x^2 - a^2} \cdot 1 \, dx \\ &= \sqrt{x^2 - a^2} \cdot x - \int \frac{1}{2\sqrt{x^2 - a^2}} (2x) \, dx \\ &= x \sqrt{x^2 - a^2} - \int \frac{x^2}{\sqrt{x^2 - a^2}} \, dx \\ &= x \sqrt{x^2 - a^2} - \int \frac{x^2 - a^2 + a^2}{\sqrt{x^2 - a^2}} \, dx \\ &= x \sqrt{x^2 - a^2} - \int \sqrt{x^2 - a^2} \, dx - a^2 \int \frac{dx}{\sqrt{x^2 - a^2}} \\ 2I &= x \sqrt{x^2 - a^2} - a^2 \int \frac{dx}{\sqrt{x^2 - a^2}} \dots\dots\dots (1) \end{aligned}$$

We know that

$$\int \frac{dx}{\sqrt{x^2 - a^2}} \, dx = \ln |x + \sqrt{x^2 - a^2}| + c$$

Put it in (1)

$$I = \frac{x \sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \ln |x + \sqrt{x^2 - a^2}| + c$$

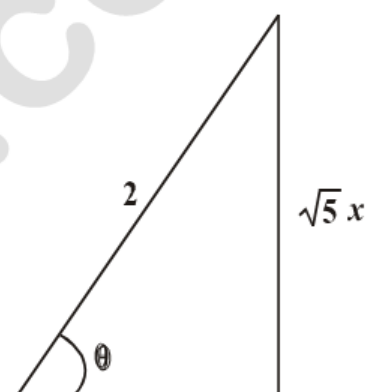
$$\begin{aligned} (iii) \quad &\int \sqrt{4 - 5x^2} \, dx \\ &= \int \sqrt{5 \left(\frac{4}{5} - x^2 \right)} \, dx \\ &= \sqrt{5} \int \sqrt{\left(\frac{2}{\sqrt{5}} \right)^2 - x^2} \, dx \end{aligned}$$

Put

$$\begin{aligned} x &= \frac{2}{\sqrt{5}} \sin \theta \quad \Rightarrow \quad \sin \theta = \frac{\sqrt{5}x}{2} \\ dx &= \frac{2}{\sqrt{5}} \cos \theta \, d\theta \quad \Rightarrow \quad \theta = \sin^{-1} \left(\frac{\sqrt{5}x}{2} \right) \end{aligned}$$



$$\begin{aligned}
 &= \sqrt{5} \int \sqrt{\left(\frac{2}{\sqrt{5}}\right)^2 - \left(\frac{2}{\sqrt{5}} \sin \theta\right)^2} \cdot \frac{2}{\sqrt{5}} \cos \theta \, d\theta \\
 &= \frac{2\sqrt{5}}{\sqrt{5}} \int \sqrt{\frac{4}{5} - \frac{4}{5} \sin^2 \theta} \cdot \cos \theta \, d\theta \\
 &= 2 \int \sqrt{\frac{4}{5} (1 - \sin^2 \theta)} \cos \theta \, d\theta \\
 &= 2 \left(\frac{2}{\sqrt{5}}\right) \int \sqrt{\cos^2 \theta} \cdot \cos \theta \, d\theta \\
 &= \frac{4}{\sqrt{5}} \int \cos \theta \cdot \cos \theta \, d\theta \\
 &= \frac{4}{\sqrt{5}} \int \cos^2 \theta \, d\theta \\
 &= \frac{4}{\sqrt{5}} \int \frac{1 + \cos 2\theta}{2} \, d\theta \\
 &= \frac{2}{\sqrt{5}} \int d\theta + \frac{2}{\sqrt{5}} \int \cos 2\theta \, d\theta \\
 &= \frac{2}{\sqrt{5}} \theta + \frac{2}{\sqrt{5}} \cdot \frac{\sin 2\theta}{2} + c \\
 &= \frac{2}{\sqrt{5}} \sin^{-1} \left(\frac{\sqrt{5}x}{2} \right) + \frac{1}{\sqrt{5}} \cdot 2 \sin \theta \cos \theta + c \\
 &= \frac{2}{\sqrt{5}} \sin^{-1} \left(\frac{\sqrt{5}x}{2} \right) + \frac{2}{\sqrt{5}} \cdot \frac{\sqrt{5}x}{2} \cdot \frac{\sqrt{4-5x^2}}{2} + c \\
 &= \frac{2}{\sqrt{5}} \sin^{-1} \left(\frac{\sqrt{5}x}{2} \right) + \frac{x}{2} \sqrt{4-5x^2} + c \text{ Ans.}
 \end{aligned}$$



$$\begin{aligned}
 4 &= B^2 + 5x^2 \\
 B^2 &= 4 - 5x^2 \\
 B &= \sqrt{4 - 5x^2}
 \end{aligned}$$

Alternative Method

$$\begin{aligned}
 I &= \int \sqrt{4 - 5x^2} \cdot 1 \cdot dx \\
 \therefore \sqrt{4 - 5x^2} &\rightarrow 1^{\text{st}} \text{ function} \\
 1 &\rightarrow 2^{\text{nd}} \text{ function} \\
 &= \sqrt{4 - 5x^2} \int 1 \, dx - \left[\int 1 \, dx \frac{d}{dx} \sqrt{4 - 5x^2} \right]
 \end{aligned}$$



$$\begin{aligned}
&= \sqrt{4-5x^2} \cdot x \cdot - \int x \cdot \frac{1}{2\sqrt{4-5x^2}} (-10x) dx \\
&= x \cdot \sqrt{4-5x^2} - \int \frac{-5x^2}{\sqrt{4-5x^2}} dx \\
&= x \sqrt{4-5x^2} - \int \frac{-4+4-5x^2}{\sqrt{4-5x^2}} dx \\
&= x \sqrt{4-5x^2} + 4 \int \frac{1}{\sqrt{4-5x^2}} dx - \int \sqrt{4-5x^2} dx \\
&= x \sqrt{4-5x^2} + 4 \int \frac{1}{\sqrt{4-5x^2}} dx - I \\
2I &= x \sqrt{4-5x^2} + \frac{4}{\sqrt{5}} \cdot \sin^{-1} \left(\frac{\sqrt{5}x}{2} \right) + c
\end{aligned}$$

÷ by 2 we know that

$$\begin{aligned}
&\int \frac{1}{\sqrt{4-5x^2}} dx = \frac{1}{\sqrt{5}} \sin^{-1} \left(\frac{\sqrt{5}x}{2} \right) + c \\
I &= \frac{x\sqrt{4-5x^2}}{2} + \frac{2}{\sqrt{5}} \sin^{-1} \left(\frac{\sqrt{5}x}{2} \right) + c
\end{aligned}$$

(iv) $\int \sqrt{3-4x^2} dx$ (Guj. Board 2005)

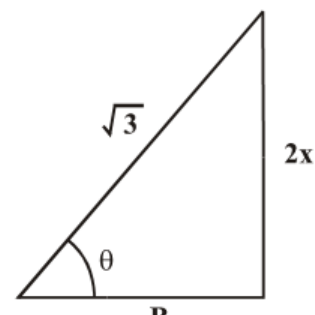
$$\begin{aligned}
&= \int \sqrt{4 \left(\frac{3}{4} - x^2 \right)} dx \\
&= 2 \int \sqrt{\left(\frac{\sqrt{3}}{2} \right)^2 - x^2} dx
\end{aligned}$$

Put

$$\begin{aligned}
x &= \frac{\sqrt{3}}{2} \sin \theta \Rightarrow \sin \theta = \frac{2x}{\sqrt{3}} \\
dx &= \frac{\sqrt{3}}{2} \cos \theta \Rightarrow \theta = \sin^{-1} \left(\frac{2x}{\sqrt{3}} \right) \\
&= 2 \sqrt{\left(\frac{\sqrt{3}}{2} \right)^2 - \left(\frac{\sqrt{3}}{2} \sin \theta \right)^2} \cdot \frac{\sqrt{3}}{2} \cos \theta d\theta \\
&= \frac{2\sqrt{3}}{2} \int \sqrt{\frac{3}{4} - \frac{3}{4} \sin^2 \theta} \cdot \cos \theta d\theta \\
&= \sqrt{3} \int \sqrt{\frac{3}{4} (1 - \sin^2 \theta)} \cos \theta d\theta
\end{aligned}$$



$$\begin{aligned}
&= \sqrt{3} \left(\frac{\sqrt{3}}{2} \right) \int \sqrt{\cos^2 \theta} \cdot \cos \theta d\theta \\
&= \frac{3}{2} \int \cos \theta \cdot \cos \theta d\theta \\
&= \frac{3}{2} \int \cos^2 \theta d\theta \\
&= \frac{3}{2} \int \frac{1 + \cos 2\theta}{2} d\theta \\
&= \frac{3}{4} \int d\theta + \frac{3}{4} \int \cos 2\theta d\theta \\
&= \frac{3}{4} \theta + \frac{3}{4} \cdot \frac{\sin 2\theta}{2} + c \\
&= \frac{3}{4} \theta + \frac{3}{8} \cdot 2 \sin \theta \cos \theta + c \\
&= \frac{3}{4} \sin^{-1} \left(\frac{2x}{\sqrt{3}} \right) + \frac{3}{4} \cdot \frac{2x}{\sqrt{3}} \cdot \frac{\sqrt{3-4x^2}}{\sqrt{3}} + c \\
&= \frac{3}{4} \sin^{-1} \left(\frac{2x}{\sqrt{3}} \right) + \frac{x}{2} \sqrt{3-4x^2} + c
\end{aligned}$$



$$3 = B^2 + 4x^2$$

$$B^2 = 3 - 4x^2$$

$$B = \sqrt{3 - 4x^2}$$

Alternative Method

$$I = \int \sqrt{3-4x^2} dx$$

Let

$$\sqrt{3-4x^2} \rightarrow 1^{\text{st}} \text{ function}$$

$$1 \rightarrow 2^{\text{nd}} \text{ function}$$

$$= \sqrt{3-4x^2} \cdot \int 1 dx - \int \left[\int 1 dx \cdot \frac{d}{dx} \sqrt{3-4x^2} \right] dx$$

$$= \sqrt{3-4x^2} \cdot x - \int x \cdot \frac{1}{2\sqrt{3-4x^2}} (-8x) dx$$

$$= \sqrt{3-4x^2} \cdot x - \int \frac{-4x^2}{\sqrt{3-4x^2}} dx$$

$$= \sqrt{3-4x^2} \cdot x - \int \frac{-3 + 3 - 4x^2}{\sqrt{3-4x^2}} dx$$

$$= x \cdot \sqrt{3-4x^2} + 3 \int \frac{1}{\sqrt{3-4x^2}} dx - \int \sqrt{3-4x^2} dx$$



$$= x \cdot \sqrt{3-4x^2} + 3 \int \frac{1}{\sqrt{3-4x^2}} dx - I$$

$$I + I = x \sqrt{3-4x^2} + 3 \int \frac{1}{\sqrt{3-4x^2}} dx$$

We know that

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right)$$

$$= x \sqrt{3-4x^2} + \frac{3}{2} \sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + c$$

$$I = \frac{x \sqrt{3-4x^2}}{2} + \frac{3}{4} \sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + c$$

$$(v) \int \sqrt{x^2+4} \, dx$$

$$\int \sqrt{x^2+(2)^2} \, dx$$

Put

$$x = 2 \tan \theta \Rightarrow \tan \theta = \frac{x}{2}$$

$$dx = 2 \sec^2 \theta d\theta$$

$$= \int \sqrt{4 \tan^2 \theta + 4} \cdot 2 \sec^2 \theta d\theta$$

$$= \int \sqrt{4(\tan^2 \theta + 1)} \cdot 2 \sec^2 \theta d\theta$$

$$= \int 2 \sqrt{\sec^2 \theta} \cdot 2 \sec^2 \theta d\theta$$

$$= 4 \int \sec \theta \cdot \sec^2 \theta d\theta$$

$$= 4 \int \sec^3 \theta d\theta$$

$$= 4 I \quad (1)$$

$$I = \int \sec^3 \theta d\theta$$

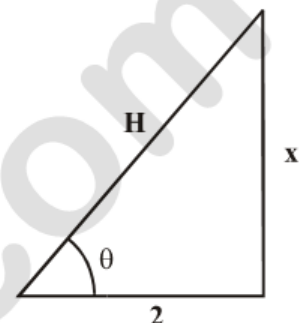
$$= \int \sec \theta \cdot \sec^2 \theta d\theta$$

$$= \sec \theta \int \sec^2 \theta d\theta - \int \left[\int \sec^2 \theta d\theta \cdot \frac{d}{d\theta}(\sec \theta) \right] d\theta$$

$$= \sec \theta \cdot \tan \theta - \int \tan \theta \cdot \sec \theta \tan \theta d\theta$$



$$\begin{aligned}
 &= \sec\theta \tan\theta - \int \sec\theta \tan^2\theta \, d\theta \\
 &= \sec\theta \tan\theta - \int \sec\theta (\sec^2\theta - 1) \, d\theta \\
 &= \sec\theta \tan\theta - \int \sec^3\theta \, d\theta + \int \sec\theta \, d\theta \\
 I &= \sec\theta \tan\theta - I + \ln |\sec\theta + \tan\theta| + c_1 \\
 2I &= \sec\theta \tan\theta + \ln |\sec\theta + \tan\theta| + c_1 \\
 \therefore &\text{From equation (1)} \\
 &= 4I \\
 &= 2(2I) \\
 &= 2[\sec\theta \tan\theta + \ln |\sec\theta + \tan\theta|] + c_1 \\
 &= 2 \frac{\sqrt{4+x^2}}{2} \cdot \frac{x}{2} + 2 \ln \left| \frac{\sqrt{4+x^2}}{2} + \frac{x}{2} \right| + c_1 \\
 &= \frac{x\sqrt{4+x^2}}{2} + 2 \ln \left| \frac{\sqrt{4+x^2} + x}{2} \right| + c_1 \\
 &= \frac{x\sqrt{4+x^2}}{2} + 2 \ln |\sqrt{4+x^2} + x| - 2 \ln 2 + c_1 \\
 &= \frac{x\sqrt{4+x^2}}{2} + 2 \ln |\sqrt{4+x^2} + x| + c \quad \text{where } c = -2 \ln 2 + c_1
 \end{aligned}$$



$$\begin{aligned}
 H^2 &= 4 + x^2 \\
 H &= \sqrt{4 + x^2}
 \end{aligned}$$

Alternative Method

$$\begin{aligned}
 I &= \int \sqrt{x^2 + 4} \, dx \\
 \text{Let } \sqrt{x^2 + 4} &\rightarrow 1^{\text{st}} \text{ function} \\
 1 &\rightarrow 2^{\text{nd}} \text{ function} \\
 I &= \sqrt{x^2 + 4} \cdot \int 1 \, dx - \int \left[\int 1 \, dx \cdot \frac{d}{dx} \sqrt{x^2 + 4} \right] dx \\
 &= x \cdot \sqrt{x^2 + 4} - \int x \cdot \frac{1}{2\sqrt{x^2 + 4}} (2x) \, dx \\
 &= x \cdot \sqrt{x^2 + 4} - \int \frac{x^2}{\sqrt{x^2 + 4}} \, dx \\
 &= x \cdot \sqrt{x^2 + 4} - \int \frac{-4 + x^2 + 4}{\sqrt{x^2 + 4}} \, dx
 \end{aligned}$$



$$= x \cdot \sqrt{x^2 + 4} + 4 \int \frac{1}{\sqrt{x^2 + 4}} dx - \int \sqrt{x^2 + 4} dx$$

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$$= x \cdot \sqrt{x^2 + 4} + 4 \int \frac{1}{\sqrt{x^2 + 4}} dx - I$$

$$2I = x \cdot \sqrt{x^2 + 4} + 4 \int \frac{1}{\sqrt{x^2 + 4}} dx$$

$$I = \frac{x\sqrt{x^2 + 4}}{2} + 2 \int \frac{1}{\sqrt{x^2 + 4}} dx$$

$$I = \frac{x\sqrt{x^2 + 4}}{2} + 2 \ln |x + \sqrt{x^2 + 4}| + c$$

$$\therefore \int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln |x + \sqrt{x^2 + a^2}| + c \quad \text{Ans.}$$

$$(vi) \int x^2 e^{ax} dx$$

$$= x^2 \int e^{ax} dx - \int \left[\int e^{ax} dx \cdot \frac{d}{dx} (x^2) \right] dx$$

$$= x^2 \frac{e^{ax}}{a} - \int \frac{e^{ax}}{a} \cdot 2x dx$$

$$= \frac{x^2 e^{ax}}{a} - \frac{2}{a} \int x e^{ax} dx$$

$$= \frac{x^2 e^{ax}}{a} - \frac{2}{a} \left[x \int e^{ax} dx - \int \left(\int e^{ax} dx \cdot \frac{d}{dx} (x) \right) dx \right]$$

$$= \frac{x^2 e^{ax}}{a} - \frac{2}{a} x \frac{e^{ax}}{a} + \frac{2}{a} \int \frac{e^{ax}}{a} \cdot 1 \cdot dx$$

$$= \frac{x^2 e^{ax}}{a} - \frac{2x e^{ax}}{a^2} + \frac{2}{a^2} \int e^{ax} dx$$

$$= \frac{x^2 e^{ax}}{a} - \frac{2x e^{ax}}{a^2} + \frac{2}{a^2} \frac{e^{ax}}{a} + c$$

$$= \frac{e^{ax}}{a} \left(x^2 - \frac{2x}{a} + \frac{2}{a^2} \right) + c \quad \text{Ans.}$$



Q.5 Evaluate the following integrals.

- (i) $\int e^x \left(\frac{1}{x} + \ln x \right) dx$ (ii) $\int e^x (\cos x + \sin x) dx$
- (iii) $\int e^{ax} \left[a \sec^{-1} x + \frac{1}{x\sqrt{x^2-1}} \right] dx$ (iv) $\int e^{3x} \left(\frac{3\sin x - \cos x}{\sin^2 x} \right) dx$
- (v) $\int e^{2x} [-\sin x + 2\cos x] dx$ (vi) $\int \frac{x e^x}{(1+x)^2} dx$
- (vii) $\int e^{-x} (\cos x - \sin x) dx$ (L.B 2009) (S) (viii) $\int \frac{e^{\tan^{-1} x}}{1+x^2} dx$
- (ix) $\int \frac{2x}{1-\sin x} dx$ (x) $\int \frac{e^x (1+x)}{(2+x)^2} dx$
- (xi) $\int \left(\frac{1-\sin x}{1-\cos x} \right) e^x dx$

Solution:

(i) $\int e^x \left(\frac{1}{x} + \ln x \right) dx$
 $= \int e^x \left(\ln x + \frac{1}{x} \right) dx$

$$\because f(x) = \ln x$$

$$f'(x) = \frac{1}{x}$$

By using the formula

$$\int e^{ax} [a f(x) + f'(x)] dx = e^{ax} f(x) + c$$

$$= e^x \ln x + c \quad \text{Ans.}$$

(ii) $\int e^x (\cos x + \sin x) dx$

By using the formula

$$\int e^{ax} [a f(x) + f'(x)] dx = e^{ax} f(x) + c$$

$$= e^x \sin x + c \quad \text{Ans.}$$

$$\because f(x) = \sin x$$

$$f'(x) = \cos x$$

(iii) $\int e^{ax} \left[a \sec^{-1} x + \frac{1}{x\sqrt{x^2-1}} \right] dx$ (Lhr. Board 2009)

By using the formula

$$\int e^{ax} [a f(x) + f'(x)] dx = e^{ax} f(x) + c$$

$$= e^{ax} \sec^{-1} x + c \quad \text{Ans.}$$

$$\because f(x) = \sec^{-1} x$$

$$f'(x) = \frac{1}{x\sqrt{x^2-1}}$$



$$\begin{aligned}
 \text{(iv)} \quad & \int e^{3x} \left(\frac{3 \sin x - \cos x}{\sin^2 x} \right) dx \\
 &= \int e^{3x} \left(\frac{3 \sin x}{\sin^2 x} - \frac{\cos x}{\sin^2 x} \right) dx \\
 &= \int e^{3x} \left(\frac{3}{\sin x} - \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} \right) dx \\
 &= \int e^{3x} (3 \operatorname{cosec} x - \operatorname{cosec} x \cot x) dx
 \end{aligned}$$

By using the formula

$$\begin{aligned}
 \int e^{ax} [a f(x) + f'(x)] dx &= e^{ax} f(x) + c \\
 &= e^{3x} \operatorname{cosec} x + c \quad \text{Ans.}
 \end{aligned}$$

$$\begin{aligned}
 \therefore f(x) &= \operatorname{cosec} x \\
 f'(x) &= -\cos x \cot x
 \end{aligned}$$

$$\text{(v)} \quad \int e^{2x} [-\sin x + 2 \cos x] dx$$

By using the formula

$$\begin{aligned}
 \int e^{ax} [a f(x) + f'(x)] dx &= e^{ax} f(x) + c \\
 &= e^{2x} \cos x + c \quad \text{Ans.}
 \end{aligned}$$

$$\begin{aligned}
 \therefore f(x) &= \cos x \\
 f'(x) &= -\sin x
 \end{aligned}$$

$$\text{(vi)} \quad \int \frac{x e^x}{(1+x)^2} dx \quad \text{(Guj. Board 2007)}$$

$$\begin{aligned}
 &= \int \frac{(1+x-1)e^x}{(1+x)^2} dx \\
 &= \int e^x \left[\frac{1+x}{(1+x)^2} - \frac{1}{(1+x)^2} \right] dx \\
 &= \int e^x \left[\frac{1}{1+x} - \frac{1}{(1+x)^2} \right] dx
 \end{aligned}$$

$$\begin{aligned}
 \therefore f(x) &= \frac{1}{1+x} \\
 f'(x) &= -\frac{1}{(1+x)^2}
 \end{aligned}$$

By using the formula

$$\begin{aligned}
 \int e^{ax} [a f(x) + f'(x)] dx &= e^{ax} f(x) + c \\
 &= \frac{e^x}{1+x} + c \quad \text{Ans.}
 \end{aligned}$$

(vii)

$$\int e^{-x} (\cos x - \sin x) dx$$

By using the formula

$$\begin{aligned}
 e^{ax} [a f(x) + f'(x)] dx &= e^{ax} f(x) + c \\
 &= e^{-x} \sin x + c
 \end{aligned}$$

$$\int e^{-x} (-\sin x + \cos x) dx$$

$$\begin{aligned}
 f(x) &= \sin x \\
 f'(x) &= \cos x
 \end{aligned}$$



$$\begin{aligned}
 \text{(viii)} \quad & \int \frac{e^{m \tan^{-1} x}}{1+x^2} dx \\
 &= \frac{1}{m} \int e^{m \tan^{-1} x} \frac{m}{1+x^2} dx \\
 &= \frac{1}{m} e^{m \tan^{-1} x} + c \quad \text{Ans.}
 \end{aligned}$$

Alternative Method

$$\text{Let } t = \tan^{-1} x$$

$$dt = \frac{1}{1+x^2} dx$$

$$\begin{aligned}
 \therefore \int \frac{e^{m \tan^{-1} x}}{1+x^2} dx &= \int e^{mt} dt \\
 &= \frac{e^{mt}}{m} + c \\
 &= \frac{1}{m} e^{m \tan^{-1} x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(ix)} \quad & \int \frac{2x}{1 - \sin x} dx \\
 &= \int \frac{2x}{1 - \cos\left(\frac{\pi}{2} - x\right)} dx \\
 &= \int \frac{2x}{2 \sin^2\left(\frac{\pi}{4} - \frac{x}{2}\right)} dx \quad \left(\begin{array}{l} \because \cos 2x = 1 - 2 \sin^2 x \\ \cos x = 1 - 2 \sin^2 \frac{x}{2} \\ 2 \sin^2 \frac{x}{2} = 1 - \cos x \end{array} \right) \\
 &= \int x \operatorname{cosec}^2\left(\frac{\pi}{4} - \frac{x}{2}\right) dx \\
 &= x \int \operatorname{cosec}^2\left(\frac{\pi}{4} - \frac{x}{2}\right) dx - \int \left[\int \operatorname{cosec}^2\left(\frac{\pi}{4} - \frac{x}{2}\right) dx \cdot \frac{d}{dx}(x) \right] dx \\
 &= x \cdot \frac{-\cot\left(\frac{\pi}{4} - \frac{x}{2}\right)}{\frac{-1}{2}} - \int \frac{-\cot\left(\frac{\pi}{4} - \frac{x}{2}\right)}{\frac{-1}{2}} \cdot 1 dx \\
 &= 2x \cot\left(\frac{\pi}{4} - \frac{x}{2}\right) - 2 \int \cot\left(\frac{\pi}{4} - \frac{x}{2}\right) dx
 \end{aligned}$$

$$\begin{array}{l}
 \because x \rightarrow 1^{\text{st}} \text{ function} \\
 \operatorname{cosec}^2\left(\frac{\pi}{4} - \frac{x}{2}\right) \rightarrow 2^{\text{nd}} \\
 \text{function}
 \end{array}$$



$$\begin{aligned}
 &= 2x \cot\left(\frac{\pi}{4} - \frac{x}{2}\right) - \frac{2 \ln \left| \sin\left(\frac{\pi}{4} - \frac{x}{2}\right) \right|}{-\frac{1}{2}} + c \\
 &= 2x \cot\left(\frac{\pi}{4} - \frac{x}{2}\right) + 4 \ln \left| \sin\left(\frac{\pi}{4} - \frac{x}{2}\right) \right| + c \quad \text{Ans.}
 \end{aligned}$$

$$\begin{aligned}
 \text{(x)} \quad &\int \frac{e^x (1+x)}{(2+x)^2} dx \\
 &= \int \frac{e^x (1+x+1-1)}{(2+x)^2} dx \\
 &= \int \frac{e^x (2+x-1)}{(2+x)^2} dx \\
 &= \int e^x \left[\frac{2+x}{(2+x)^2} - \frac{1}{(2+x)^2} \right] dx \\
 &= \int e^x \left[\frac{1}{2+x} - \frac{1}{(2+x)^2} \right] dx
 \end{aligned}$$

$$\begin{aligned}
 \therefore f(x) &= \frac{1}{2+x} \\
 f'(x) &= \frac{-1}{(2+x)^2}
 \end{aligned}$$

By using the formula

$$\begin{aligned}
 &\int e^{ax} [a f(x) + f'(x)] dx = e^{ax} f(x) + c \\
 &= \frac{e^x}{2+x} + c \quad \text{Ans.}
 \end{aligned}$$

$$\begin{aligned}
 \text{(xi)} \quad &\int \left(\frac{1 - \sin x}{1 - \cos x} \right) e^x dx \\
 &= \int e^x \left(\frac{1 - 2\sin \frac{x}{2} \cos \frac{x}{2}}{2\sin^2 \frac{x}{2}} \right) dx \\
 &= \int e^x \left(\frac{1}{2\sin^2 \frac{x}{2}} - \frac{2\sin \frac{x}{2} \cos \frac{x}{2}}{2\sin^2 \frac{x}{2}} \right) dx \\
 &= \int \left(\frac{1}{2} \operatorname{cosec}^2 \frac{x}{2} - \frac{\cos \frac{x}{2}}{\sin \frac{x}{2}} \right) e^x dx \\
 &= - \int \left(\frac{-1}{2} \operatorname{cosec}^2 \frac{x}{2} + \cot \frac{x}{2} \right) e^x dx
 \end{aligned}$$

$$\begin{aligned}
 \therefore f(x) &= \cot \frac{x}{2} \\
 f'(x) &= -\operatorname{csc}^2 \frac{x}{2} \left(\frac{1}{2} \right) \\
 f'(x) &= -\frac{1}{2} \operatorname{csc}^2 \left(\frac{x}{2} \right)
 \end{aligned}$$



By using the formula

$$\int e^{ax} [a f(x) + f'(x)] dx = e^{ax} f(x) + c$$
$$= -e^x \cot \frac{x}{2} + c \quad \text{Ans.}$$

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EXERCISE 3.5

Q.1 $\int \frac{3x+1}{x^2-x-6} dx$

Solution:

$$\begin{aligned} & \int \frac{3x+1}{x^2-x-6} dx \\ &= \int \frac{3x+1}{x^2-3x+2x-6} dx \\ &= \int \frac{3x+1}{x(x-3)+2(x-3)} dx \\ &= \int \frac{3x+1}{(x+2)(x-3)} dx \end{aligned}$$

Let

$$\frac{3x+1}{(x+2)(x-3)} = \frac{A}{x+2} + \frac{B}{x-3} \quad \text{--- (1)}$$

Where A & B are constant of partial fractions which are to be determined.

Multiplying $(x+2)(x-3)$ on both sides in (1)

$$3x+1 = A(x-3) + B(x+2) \quad \text{--- (2)}$$

To find A

Put $x+2=0$

$x = -2$ in (2)

$$3(-2)+1 = A(-2-3)$$

$$-6+1 = A(-5)$$

$$-5 = -5A$$

$$A = \frac{-5}{-5}$$

$A = 1$

To find B

Put $x-3=0$



$$\begin{aligned}
 x &= 3 \text{ in (2)} \\
 3(3) + 1 &= B(3 + 2) \\
 9 + 1 &= 5B \\
 5B &= 10 \\
 B &= \frac{10}{5}
 \end{aligned}$$

∴ From eq. (1)

$$\begin{aligned}
 \frac{3x+1}{(x+2)(x-3)} &= \frac{1}{x+2} + \frac{2}{x-3} \\
 \int \frac{3x+1}{(x+2)(x-3)} dx &= \int \frac{dx}{x+2} + 2 \int \frac{dx}{x-3} \\
 &= \boxed{\ln|x+2| + 2\ln|x-3| + c} \quad \text{Ans.}
 \end{aligned}$$

Q.2 $\int \frac{5x+8}{(x+3)(2x-1)} dx$

Solution:

$$\int \frac{5x+8}{(x+3)(2x-1)} dx$$

Let

$$\frac{5x+8}{(x+3)(2x-1)} = \frac{A}{x+3} + \frac{B}{2x-1} \quad (1)$$

Where A & B are constant of partial fractions which are to be determined.

Multiplying $(x+3)(2x-1)$ on both sides in (1)

$$5x+8 = A(2x-1) + B(x+3) \quad (2)$$

To find A

$$\text{Put } x+3=0$$

$$x = -3 \text{ in (2)}$$

$$5(-3) + 8 = A[2(-3) - 1]$$

$$-15 + 8 = A(-6 - 1)$$

$$-7 = -7A$$

$$A = \frac{-7}{-7}$$

$$\boxed{A = 1}$$

To find B

$$\text{Put } 2x-1 = 0$$

$$2x = 1$$



$$x = \frac{1}{2} \ln(2)$$

$$5\left(\frac{1}{2}\right) + 8 = B\left(\frac{1}{2} + 3\right)$$

$$\frac{5}{2} + 8 = B\left(\frac{1+6}{2}\right)$$

$$\frac{5+16}{2} = \left(\frac{7}{2}\right) B$$

$$\frac{21}{2} = \frac{7}{2} B$$

$$\frac{21}{2} \times \frac{2}{7} = B$$

$$\boxed{B = 3}$$

∴ From eq. (1)

$$\frac{5x+8}{(x+3)(2x-1)} = \frac{1}{x+3} + \frac{3}{2x-1}$$

Integrate

$$\int \frac{5x+8}{(x+3)(2x-1)} dx = \int \frac{dx}{x+3} + \frac{3}{2} \int \frac{2dx}{2x-1}$$

$$= \boxed{\ln|x+3| + \frac{3}{2} \ln|2x-1| + c} \quad \text{Ans.}$$

Q.3 $\int \frac{x^2+3x-34}{x^2+2x-15} dx$ (Guj. Board 2006)

Solution:

$$\int \frac{x^2+3x-34}{x^2+2x-15} dx$$

$$= \int \left(1 + \frac{x-19}{x^2+2x-15}\right) dx$$

$$= \int dx + \int \frac{x-19}{x^2+5x-3x-15} dx$$

$$= x + \int \frac{x-19}{x(x+5)-3(x+5)} dx$$

$$= x + \int \frac{x-19}{(x-3)(x+5)} dx$$

$$\frac{1}{x^2+2x-15} = \frac{1}{(x+5)(x-3)} = \frac{A}{x+5} + \frac{B}{x-3}$$

$$1 = A(x-3) + B(x+5)$$

$$1 = Ax - 3A + Bx + 5B$$

$$1 = (A+B)x + (-3A+5B)$$

$$\begin{cases} A+B=0 \\ -3A+5B=1 \end{cases}$$

$$\begin{aligned} A &= -B \\ -3(-B)+5B &= 1 \\ 3B+5B &= 1 \\ 8B &= 1 \\ B &= \frac{1}{8} \\ A &= -\frac{1}{8} \end{aligned}$$

$$\frac{1}{x^2+2x-15} = \frac{-\frac{1}{8}}{x+5} + \frac{\frac{1}{8}}{x-3}$$



$$= \frac{x+1}{(x-3)(x+5)} \quad (1)$$

$$I = \int \frac{x-19}{(x-3)(x+5)} dx$$

Let

$$\frac{x-19}{(x-3)(x+5)} = \frac{A}{x-3} + \frac{B}{x+5} \quad (2)$$

Where A & B are constant of potential fractions which are to be determined.

Multiplying $(x-3)(x+5)$ on both sides in (2)

$$x-19 = A(x+5) + B(x-3) \quad (3)$$

To find A

$$\text{Put } x-3 = 0$$

$$x = 3 \text{ in (3)}$$

$$3-19 = A(3+5)$$

$$-16 = 8A$$

$$A = \frac{-16}{8}$$

$$\boxed{A = -2}$$

To find B

$$\text{Put } x+5 = 0$$

$$x = -5 \text{ in (3)}$$

$$-5-19 = B(-5-3)$$

$$-24 = -8B$$

$$B = \frac{-24}{-8}$$

$$\boxed{B = 3}$$

∴ From eq. (2)

$$\frac{x-19}{(x-3)(x+5)} = \frac{-2}{x-3} + \frac{3}{x+5}$$

Integrate

$$\int \frac{x-19}{(x-3)(x+5)} dx = 2 \int \frac{dx}{x-3} + 3 \int \frac{dx}{x+5}$$

$$I = -2 \ln |x-3| + 3 \ln |x+5| + c$$

∴ From eq. (1)

$$= \boxed{x - 2 \ln |x-3| + 3 \ln |x+5| + c} \quad \text{Ans.}$$



Q.4 $\int \frac{(a-b)x}{(x-a)(x-b)} dx$

Solution:

$$\int \frac{(a-b)x}{(x-a)(x-b)} dx$$

Let

$$\frac{(a-b)x}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} \quad \text{———— (1)}$$

Where A & B are constant of partial fractions which are to be determined.

Multiplying $(x-a)(x-b)$ on both sides in (1)

$$(a-b)x = A(x-b) + B(x-a) \quad \text{———— (2)}$$

To find A

Put $x - a = 0$

$$x = a \text{ in (2)}$$

$$(a-b)a = A(a-b)$$

$$\frac{(a-b)a}{a-b} = A$$

$$\boxed{A = a}$$

To find B

Put $x - b = 0$

$$x = b \text{ in eq. (2)}$$

$$(a-b)b = B(b-a)$$

$$\frac{(a-b)b}{b-a} = B$$

$$B = \frac{-(b-a)b}{b-a}$$

$$\boxed{B = -b}$$

From eq. (1)

$$\frac{(a-b)x}{(x-a)(x-b)} = \frac{a}{x-a} + \frac{-b}{x-b}$$

Integrate

$$\int \frac{(a-b)x}{(x-a)(x-b)} dx = a \int \frac{dx}{x-a} - b \int \frac{dx}{x-b}$$

$$= \boxed{a \ln |x-a| - b \ln |x-b| + c} \quad \text{Ans.}$$



Q.5 $\int \frac{3-x}{1-x-6x^2} dx$

Solution:

$$\begin{aligned} & \int \frac{3-x}{1-x-6x^2} dx \\ &= \int \frac{3-x}{1-3x+2x-6x^2} dx \\ &= \int \frac{3-x}{(1-3x)+2x(1-3x)} dx \\ &= \int \frac{3-x}{(1+2x)(1-3x)} dx \end{aligned}$$

Let

$$\frac{3-x}{(1+2x)(1-3x)} = \frac{A}{1+2x} + \frac{B}{1-3x} \quad \text{————— (1)}$$

Where A and B are constant of partial fractions which are to be determined

Multiplying $(1+2x)(1-3x)$ on both sides in (1)

$$3-x = A(1-3x) + B(1+2x) \quad \text{————— (2)}$$

To find A

Put $1+2x = 0$

$$2x = -1$$

$$x = \frac{-1}{2} \text{ in (2)}$$

$$3 - \frac{-1}{2} = A \left[1 - 3 \left(\frac{-1}{2} \right) \right]$$

$$3 + \frac{1}{2} = A \left(1 + \frac{3}{2} \right)$$

$$\frac{6+1}{2} = A \left(\frac{2+3}{2} \right)$$

$$\frac{7}{2} = \left(\frac{5}{2} \right) A$$

$$\frac{7 \times 2}{2 \times 5} = A$$

$$\boxed{A = \frac{7}{5}}$$



To find B

$$\text{Put } 1 - 3x = 0$$

$$3x = 1$$

$$x = \frac{1}{3} \text{ in (2)}$$

$$3 - \frac{1}{3} = B \left(1 + 2 \left(\frac{1}{3} \right) \right)$$

$$\frac{9-1}{3} = B \left(1 + \frac{2}{3} \right)$$

$$\frac{8}{3} = B \left(\frac{3+2}{3} \right)$$

$$\frac{8}{3} = \left(\frac{5}{3} \right) B$$

$$\frac{8}{3} \times \frac{3}{5} = B$$

$$\boxed{B = \frac{8}{5}}$$

∴ From eq. (1)

$$\frac{3-x}{(1+2x)(1-3x)} = \frac{\frac{7}{5}}{1+2x} + \frac{\frac{8}{5}}{1-3x}$$

Integrate

$$\int \frac{3-x}{(1+2x)(1-3x)} dx = \frac{7}{5} \int \frac{dx}{1+2x} + \frac{8}{5} \int \frac{dx}{1-3x}$$

$$= \frac{7}{5 \cdot 2} \int \frac{2dx}{1+2x} - \frac{8}{5 \times 3} \int \frac{-3}{1-3x} dx$$

$$= \boxed{\frac{7}{10} \ln |1+2x| - \frac{8}{5} \ln |1-3x| + c} \quad \text{Ans.}$$

Q.6 $\int \frac{2x}{x^2 - a^2} dx$

Solution:

$$\int \frac{2x}{x^2 - a^2} dx$$

$$= \int \frac{2x}{(x+a)(x-a)} dx$$



Let

$$\frac{2x}{(x+a)(x-a)} = \frac{A}{x+a} + \frac{B}{x-a} \quad (1)$$

Where A and B are constant of partial fractions which are to be determined

Multiplying $(x+a)(x-a)$ on both sides in (1)

$$2x = A(x-a) + B(x+a) \quad (2)$$

To find A

$$\text{Put } x+a = 0$$

$$x = -a \text{ in (2)}$$

$$2(-a) = A(-a-a)$$

$$-2a = 2aA$$

$$A = \frac{-2a}{-2a}$$

$$\boxed{A = 1}$$

To find B

$$\text{Put } x-a = 0$$

$$x = a \text{ in (2)}$$

$$2a = B(a+a)$$

$$2a = 2aB$$

$$B = \frac{2a}{2a}$$

$$\boxed{B = 1}$$

∴ From eq. (1)

$$\frac{2x}{(x+a)(x-a)} = \frac{1}{x+a} + \frac{1}{x-a}$$

Integrate

$$\int \frac{2x}{(x+a)(x-a)} dx = \int \frac{dx}{x+a} + \int \frac{dx}{x-a}$$

$$= \ln|x+a| + \ln|x-a| + c$$

Ans.

Q.7 $\int \frac{1}{6x^2 - 5x - 4} dx$

Solution:

$$\int \frac{1}{6x^2 - 5x - 4} dx$$



$$\begin{aligned}
 &= \int \frac{1}{x^2 + 8x - 3x - 4} dx \\
 &= \int \frac{1}{2x(3x+4) - 1(3x+4)} dx \\
 &= \int \frac{1}{(2x-1)(3x+4)} dx
 \end{aligned}$$

Let

$$\frac{1}{(2x-1)(3x+4)} = \frac{A}{2x-1} + \frac{B}{3x+4} \quad \text{————— (1)}$$

Where A and B are constant of partial fractions which are to be determined.

Multiplying $(2x-1)(3x+4)$ on both sides in (1)

$$1 = A(3x+4) + B(2x-1) \quad \text{————— (2)}$$

To find A

$$\begin{aligned}
 \text{Put } 2x-1 &= 0 \\
 2x &= 1 \\
 x &= \frac{1}{2} \text{ in eq. (2)}
 \end{aligned}$$

$$1 = A\left(\frac{3}{2} + 4\right)$$

$$1 = A\left(\frac{3+8}{2}\right)$$

$$2 = A(11)$$

$$A = \frac{2}{11}$$

To find B

$$\begin{aligned}
 \text{Put } 3x+4 &= 0 \\
 3x &= -4 \\
 x &= \frac{-4}{3} \text{ in eq. (2)}
 \end{aligned}$$

$$1 = B\left[2\left(\frac{-4}{3}\right) - 1\right]$$

$$1 = B\left(\frac{-8}{3} - 1\right)$$

$$1 = B\left(\frac{-8-3}{3}\right)$$



$$3 = -11B$$

$$B = \frac{-3}{11}$$

∴ From eq. (1)

$$\begin{aligned} \frac{1}{(2x-1)(3x+4)} &= \frac{\frac{2}{11}}{2x-1} + \frac{\frac{-3}{11}}{3x+4} \\ \int \frac{dx}{(2x-1)(3x+4)} &= \frac{1}{11} \int \frac{2dx}{2x-1} - \frac{1}{11} \int \frac{3}{3x+4} dx \\ &= \left[\frac{1}{11} \ln |2x-1| - \frac{1}{11} \ln |3x+4| + c \right] \quad \text{Ans.} \end{aligned}$$

Q.8 $\int \frac{2x^3 - 3x^2 - x - 7}{2x^2 - 3x - 2} dx$

Solution:

$$\begin{aligned} &\int \frac{2x^3 - 3x^2 - x - 7}{2x^2 - 3x - 2} dx \\ &= \int \left(x + \frac{x-7}{2x^2 - 3x - 2} \right) dx \quad \because 2x^2 - 3x - 2 \overline{) 2x^3 - 3x^2 - x - 7} \\ &= \int x dx + \int \frac{x-7}{2x^2 - 4x + x - 2} dx \\ &= \frac{x^2}{2} + \int \frac{x-7}{2x(x-2) + 1(x-2)} dx \\ &= \frac{x^2}{2} + \int \frac{x-7}{(2x+1)(x-2)} dx \\ &= \frac{x^2}{2} + I \quad \text{--- (1)} \end{aligned}$$

Where

$$\begin{aligned} I &= \int \frac{x-7}{(2x+1)(x-2)} dx \\ \frac{x-7}{(2x+1)(x-2)} &= \frac{A}{2x+1} + \frac{B}{x-2} \quad \text{--- (2)} \end{aligned}$$

Where A & B are constant of partial fractions which are to be determined.

Multiplying $(2x+1)(x-2)$ on both sides in (2)



$$x - 7 = A(x - 2) + B(2x + 1) \text{ ————— (3)}$$

To find A

$$\begin{aligned} \text{Put } 2x + 1 &= 0 \\ 2x &= -1 \\ x &= \frac{-1}{2} \text{ in (3)} \end{aligned}$$

$$\frac{-1}{2} - 7 = A \left(\frac{-1}{2} - 2 \right)$$

$$\frac{-1 - 14}{2} = A \left(\frac{-1 - 4}{2} \right)$$

$$\frac{-15}{2} = A \left(\frac{-5}{2} \right)$$

$$\frac{-15}{2} \times \frac{-2}{-5} = A$$

$$\boxed{A = 3}$$

To find B

$$\begin{aligned} \text{Put } x - 2 &= 0 \\ x &= 2 \text{ in (3)} \end{aligned}$$

$$2 - 7 = B[2(2) + 1]$$

$$-5 = B(4 + 1)$$

$$-5 = 5B$$

$$B = \frac{-5}{5}$$

$$\boxed{B = -1}$$

∴ From eq. (2)

$$\frac{x - 7}{(2x + 1)(x - 2)} = \frac{3}{2x + 1} + \frac{-1}{x - 2}$$

$$\int \frac{x - 7}{(2x + 1)(x - 2)} dx = \frac{3}{2} \int \frac{2dx}{2x + 1} - \int \frac{dx}{x - 2}$$

$$= \boxed{\frac{3}{2} \ln |2x + 1| - \ln |x - 2| + c}$$

Ans.

Q.9 $\int \frac{3x^2 - 12x + 11}{(x - 1)(x - 2)(x - 3)} dx$

Solution:



$$\int \frac{3x^2 - 12x + 11}{(x-1)(x-2)(x-3)} dx$$

Let $\frac{3x^2 - 12x + 11}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$ ——— (1)

Where A, B and C are constant of partial fractions which are to be determined.

Multiplying $(x-1)(x-2)(x-3)$ on both sides in (1)

$$3x^2 - 12x + 11 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \text{ ———(2)}$$

To find A

Put $x - 1 = 0$

$$x = 1 \text{ in (2)}$$

$$3(1)^2 - 12(1) + 11 = A(1-2)(1-3)$$

$$3 - 12 + 11 = A(-1)(-2)$$

$$2 = 2A$$

$$A = \frac{2}{2}$$

$$\boxed{A = 1}$$

To find B

Put $x - 2 = 0$

$$x = 2 \text{ in (2)}$$

$$3(2)^2 - 12(2) + 11 = B(2-1)(2-3)$$

$$3(4) - 24 + 11 = B(1)(-1)$$

$$12 - 24 + 11 = -B$$

$$-1 = -B$$

$$\boxed{B = 1}$$

To find C

∴ From eq. (1)

$$\frac{3x^2 - 12x + 11}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3}$$

Integrate

$$\int \frac{3x^2 - 12x + 11}{(x-1)(x-2)(x-3)} dx = \int \frac{dx}{x-1} + \int \frac{dx}{x-2} + \int \frac{dx}{x-3}$$



$$= \boxed{\ln|x-1| + \ln|x-2| + \ln|x-3| + c} \quad \text{Ans.}$$

Q.10 $\int \frac{2x-1}{x(x-1)(x-3)} dx$

Solution:

$$\int \frac{2x-1}{x(x-1)(x-3)} dx$$

Let

$$\frac{2x-1}{x(x-1)(x-3)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-3} \quad \text{———— (1)}$$

Where A, B and C are constant of partial fractions which are to be determined.

Multiplying $x(x-1)(x-3)$ on both sides in (1)

$$2x-1 = A(x-1)(x-3) + Bx(x-3) + Cx(x-1) \quad \text{————(2)}$$

To find A

Put $x = 0$ in eq (2)

$$2(0)-1 = A(0-1)(0-3)$$

$$-1 = A(-1)(-3)$$

$$-1 = 3A$$

$$\boxed{A = \frac{-1}{3}}$$

To find B

Put $x-1 = 0$

$$x = 1 \text{ in (2)}$$

$$2(1)-1 = B(1)(1-3)$$

$$2-1 = B(-2)$$

$$1 = -2B$$

$$\boxed{B = \frac{-1}{2}}$$

To find C

Put $x-3 = 0$

$$x = 3 \text{ in (2)}$$

$$2(3)-1 = C(3)(3-1)$$

$$6-1 = C(3)(2)$$

$$5 = 6C$$



$$C = \frac{5}{6}$$

∴ From eq. (1)

$$\frac{2x-1}{x(x-1)(x-3)} = \frac{-1}{x} + \frac{-1}{x-1} + \frac{5}{x-3}$$

Integrate $\int \frac{2x-1}{x(x-1)(x-3)} dx = \frac{-1}{3} \int \frac{dx}{x} - \frac{1}{2} \int \frac{dx}{x-1} + \frac{5}{6} \int \frac{dx}{x-3}$

$$= \left[\frac{-1}{3} \ln |x| - \frac{1}{2} \ln |x-1| + \frac{5}{6} \ln |x-3| + c \right] \text{ Ans.}$$

Q.11 $\int \frac{5x^2 + 9x + 6}{(x^2 - 1)(2x + 3)} dx$

Solution:

$$\begin{aligned} & \int \frac{5x^2 + 9x + 6}{(x^2 - 1)(2x + 3)} dx \\ &= \int \frac{5x^2 + 9x + 6}{(x+1)(x-1)(2x+3)} dx \end{aligned}$$

Let

$$\frac{5x^2 + 9x + 6}{(x+1)(x-1)(2x+3)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{2x+3} \quad \text{--- (1)}$$

Where A, B and C are constant of partial fractions which are to be determined.

Multiplying $(x+1)(x-1)(2x+3)$ on both sides in (1)

$$5x^2 + 9x + 6 = A(x-1)(2x+3) + B(x+1)(2x+3) + C(x-1)(x-1) \quad \text{--- (2)}$$

To find A

Put $x + 1 = 0$

$x = -1$ in (2)

$$5(-1)^2 + 9(-1) + 6 = A(-1-1)(-2+3)$$

$$5 - 9 + 6 = A(-2)(1)$$

$$2 = -2A$$

$$A = \frac{2}{-2}$$

$$A = -1$$

To find B

Put $x - 1 = 0$

$x = 1$ in (2)



$$5(1)^2 + 9(1) + 6 = B(1+1)(2+3)$$

$$5 + 9 + 6 = B(2)(5)$$

$$20 = 10B$$

$$B = \frac{20}{10}$$

$$B = 2$$

To find C

$$\text{Put } 2x + 3 = 0$$

$$2x = -3$$

$$x = \frac{-3}{2} \text{ in (2)}$$

$$5\left(\frac{-3}{2}\right)^2 + 9\left(\frac{-3}{2}\right) + 6 = C\left(\frac{-3}{2} + 1\right)\left(\frac{-3}{2} - 1\right)$$

$$5\left(\frac{9}{4}\right) - \frac{27}{2} + 6 = C\left(\frac{-3+2}{2}\right)\left(\frac{-3-2}{2}\right)$$

$$\frac{45}{4} - \frac{27}{2} + 6 = C\left(\frac{-1}{2}\right)\left(\frac{-5}{2}\right)$$

$$\frac{45 - 54 + 24}{4} = \frac{5}{4} C$$

$$\frac{15}{4} \times \frac{4}{5} = C$$

$$C = 3$$

∴ From eq. (1)

$$\frac{5x^2 + 9x + 6}{(x+1)(x-1)(2x+3)} = \frac{-1}{x+1} + \frac{2}{x-1} + \frac{3}{2x+3}$$

Integrate

$$\int \frac{5x^2 + 9x + 6}{(x+1)(x-1)(2x+3)} dx = -\int \frac{dx}{x+1} + 1 \int \frac{dx}{x-1} + \frac{3}{2} \int \frac{2}{2x+3} dx$$

$$= \ln|x+1| + 2\ln|x-1| + \frac{3}{2} \ln|2x+3| + c \quad \text{Ans.}$$

Q.12 $\int \frac{4+7x}{(1+x)^2(2+3x)} dx$

Solution:



$$\int \frac{4 + 7x}{(1 + x)^2 (2 + 3x)} dx$$

Let

$$\frac{4 + 7x}{(2 + 3x)(1 + x)^2} = \frac{A}{2 + 3x} + \frac{B}{1 + x} + \frac{C}{(1 + x)^2} \quad \text{--- (1)}$$

Where A, B and C are constant of partial fractions which are to be determined.

Multiplying $(2 + 3x)(1 + x)^2$ on both sides in (1)

$$4 + 7x = A(1 + x)^2 + B(2 + 3x)(1 + x) + C(2 + 3x) \quad \text{--- (2)}$$

$$4 + 7x = A(1 + x^2 + 2x) + B(2 + 5x + 3x^2) + 2C + 3Cx$$

$$4 + 7x = (A + 3B)x^2 + (2A + 5B + 3C)x + (A + 2B + 2C) \quad \text{--- (3)}$$

To find A

$$\text{Put } 2 + 3x = 0$$

$$3x = -2$$

$$x = \frac{-2}{3} \quad \text{in (3)}$$

$$4 + 7\left(\frac{-2}{3}\right) = A\left(1 - \frac{2}{3}\right)^2$$

$$4 - \frac{14}{3} = A\left(\frac{3-2}{3}\right)^2$$

$$\frac{12 - 14}{3} = A\left(\frac{1}{3}\right)^2$$

$$\frac{-2}{3} = \frac{1}{9}A$$

$$\frac{-2 \times 9}{3} = A$$

$$\boxed{A = -6}$$

To find C

$$\text{Put } 1 + x = 0$$

$$x = -1 \quad \text{in (2)}$$

$$4 + 7(-1) = C(2 - 3)$$

$$4 - 7 = C(-1)$$

$$-3 = -C$$

$$\boxed{C = 3}$$

To find B comparing the coefficient of x^2 in (3)

$$A + 3B = 0$$



$$3B = -A$$

$$B = -\frac{(-6)}{3}$$

$$\boxed{B = 2}$$

∴ From eq. (1)

$$\frac{4 + 7x}{(2 + 3x)(1 + x)^2} = \frac{-6}{2 + 3x} + \frac{2}{1 + x} + \frac{3}{(1 + x)^2}$$

Integrate

$$\int \frac{(4 + 7x)}{(2 + 3x)(1 + x)^2} dx = -\frac{6}{3} \int \frac{3dx}{2 + 3x} + 2 \int \frac{dx}{1 + x} + 3 \int \frac{dx}{(1 + x)^2}$$

$$= -2 \ln |2 + 3x| + 2 \ln |1 + x| + 3 \int (1 + x)^{-2} dx$$

$$= -2 \ln |2 + 3x| + 2 \ln |1 + x| + \frac{(1 + x)^{-2+1}}{-2+1} + c$$

$$= -2 \ln |2 + 3x| + 2 \ln |1 + x| + 3 \frac{(1 + x)^{-1}}{-1} + c$$

$$= \boxed{-2 \ln |2 + 3x| + 2 \ln |1 + x| - \frac{3}{1 + x} + c} \quad \text{Ans.}$$

Q.13 $\int \frac{x^2}{(x-1)(x+1)} dx$

Solve as question # 12 solved.

Q.14 $\int \frac{1}{(x-1)(x+1)^2} dx$

Solve as question # 12 is solved.

Q.15 $\int \frac{x+4}{(x^3-3x^2+4)} dx$

Solution:

$$\int \frac{x+4}{(x^3-3x^2+4)} dx$$

-1	1	-3	0	4
		-1	4	-4
	1	-4	4	0

So

$$x^3 - 3x^2 + 4 = (x + 1)(x^2 - 4x + 4)$$



$$= (x+1)(x-2)^2$$

$$\therefore \int \frac{x+4}{(x+1)(x-2)^2}$$

Let

$$\frac{x+4}{(x+1)(x-2)^2} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} \quad \text{———— (1)}$$

Where A, B and C are constant of partial fractions which are to be determined.

Multiplying $(x+1)(x-2)^2$ on both sides in (1)

$$x+4 = A(x-2)^2 + B(x+1)(x-2) + C(x+1) \quad \text{———— (2)}$$

$$x+4 = A(x^2+4-4x) + B(x^2-x-2) + Cx+C$$

$$x+4 = Ax^2+4A-4Ax+Bx^2-Bx-2B+Cx+C$$

$$x+4 = (A+B)x^2 + (-4A-B+C)x + (4A-2B+C) \quad \text{———— (3)}$$

To find A

$$\text{Put } x+1 = 0$$

$$x = -1 \text{ in (2)}$$

$$-1+4 = A(-1-2)^2$$

$$3 = A(-3)^2$$

$$3 = 9A$$

$$A = \frac{3}{9}$$

$$\boxed{A = \frac{1}{3}}$$

To find C

$$\text{Put } x-2 = 0$$

$$x = 2 \text{ in (2)}$$

$$2+4 = C(2+1)$$

$$6 = 3C$$

$$C = \frac{6}{3}$$

$$\boxed{C = 2}$$

To find B comparing the coefficient of x^2 in (3)

$$A+B = 0$$

$$B = -A$$

$$\boxed{B = -\frac{1}{3}}$$

$\therefore$ From (1)



$$\frac{x+4}{(x+1)(x-2)^2} = \frac{\frac{1}{3}}{x+1} + \frac{2}{x-2} + \frac{\frac{-1}{3}}{(x-2)^2}$$

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Integrate

$$\begin{aligned}
 \int \frac{x+4}{(x+1)(x-2)^2} &= \frac{1}{3} \int \frac{dx}{x+1} + 2 \int \frac{dx}{x-2} - \frac{1}{3} \int \frac{dx}{(x-2)^2} \\
 &= \frac{1}{3} \ln|x+1| + 2 \ln|x-2| - \frac{1}{3} \int (x-2)^{-2} dx \\
 &= \frac{1}{3} \ln|x+1| + 2 \ln|x-2| - \frac{1}{3} \frac{(x-2)^{-2+1}}{-2+1} + c \\
 &= \frac{1}{3} \ln|x+1| + 2 \ln|x-2| - \frac{1}{3} \frac{(x-2)^{-1}}{-1} + c \\
 &= \boxed{\frac{1}{3} \ln|x+1| + 3 \ln|x-2| + \frac{1}{3(x-2)} + c} \quad \text{Ans.}
 \end{aligned}$$

Q.16 $\int \frac{x^3 - 6x^2 + 25}{(x+1)^2 (x-2)^2} dx$

Solution:

$$\int \frac{x^3 - 6x^2 + 25}{(x+1)^2 (x-2)^2} dx$$

Let $\frac{x^3 - 6x^2 + 25}{(x+1)^2 (x-2)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2}$

Where A, B, C and D are constant of partial fractions which are to be determined.

Multiply $(x+1)^2 (x-2)^2$ on both sides in (1)

$$x^3 - 6x^2 + 25 = A(x+1)(x-2)^2 + B(x-2)^2 + C(x-2)(x+1)^2 + D(x+1)^2 \quad (2)$$

$$x^3 - 6x^2 + 25 = A(x+1)(x^2+4-4x) + B(x^2+4-4x) + C(x-2)(x^2+1+2x) + D(x^2+1+2x)$$

$$x^3 - 6x^2 + 25 = A(x^3+4x-4x^2+x^2+4-4x) + Bx^2+4B-4Bx + C(x^3+x+2x^2-2x^2-2-4x) + Dx^2 + D + 2Dx$$

$$x^3 - 6x^2 + 25 = Ax^3 - 3Ax^2 + 4A + Bx^2 + 4B - 4Bx + Cx^3 - 3Cx - 2C + Dx^2 + D + 2Dx$$

$$x^3 - 6x^2 + 25 = (A+C)x^3 + (-3A+B+D)x^2 + (-4B-3C+2D)x + (4A+4B-2C+D) \quad (3)$$

To find B

Put $x+1 = 0$

$$x = -1 \text{ in (2)}$$

$$(-1)^3 - 6(-1)^2 + 25 = B(-1-2)^2$$

$$-1 - 6 + 25 = B(-3)^2$$

$$18 = 9B$$

$$B = \frac{18}{9}$$

$$\boxed{B = 2}$$



To find D

$$\text{Put } x - 2 = 0$$

$$x = 2 \text{ in (2)}$$

$$(2)^3 - 6(2)^2 + 25 = D(2 + 1)^2$$

$$8 - 6(4) + 25 = (3)^2 D$$

$$8 - 24 + 25 = 9D$$

$$9 = 9D$$

$$D = \frac{9}{9}$$

$$\boxed{D = 1}$$

To find A and C comparing the coefficients of x^3 , x^2 in (3)

$$A + C = 1 \text{ — (i)}$$

$$-3A + B + D = -6$$

$$-3A + 2 + 1 = -6$$

$$-3A + 3 = -6$$

$$-3A = -6 - 3$$

$$A = \frac{-9}{-3}$$

$$\boxed{A = 3}$$

$$\text{Put } A = 3 \text{ in eq. (1)}$$

$$A + C = 1$$

$$C = 1 - A$$

$$C = 1 - 3$$

$$\boxed{C = -2}$$

∴ From eq. (1)

$$\frac{x^3 - 6x^2 + 25}{(x + 1)^2 (x - 2)^2} = \frac{3}{x + 1} + \frac{2}{(x + 1)^2} + \frac{-2}{x - 2} + \frac{1}{(x - 2)^2}$$

Integrate

$$\int \frac{x^3 - 6x^2 + 25}{(x + 1)^2 (x - 2)^2} dx = 3 \int \frac{dx}{x + 1} + 2 \int \frac{dx}{(x + 1)^2} - 2 \int \frac{dx}{x - 2} + \int \frac{dx}{(x - 2)^2}$$

$$= 3 \ln |x + 1| + 2 \int (x + 1)^{-2} dx - 2 \ln |x - 2| + \int (x - 2)^{-2} dx$$

$$= 3 \ln |x + 1| + 2 \frac{(x + 1)^{-2+1}}{-2+1} - 2 \ln |x - 2| + \frac{(x - 2)^{-2+1}}{-2+1} + c$$



$$\begin{aligned}
 &= 3 \ln |x+1| + \frac{2(x+1)^{-1}}{-1} - 2 \ln |x-2| + \frac{(x-2)^{-1}}{-1} + c \\
 &= \boxed{3 \ln |x+1| - \frac{2}{x+1} - 2 \ln |x-2| - \frac{1}{x-2} + c} \quad \text{Ans.}
 \end{aligned}$$

Q.17 $\int \frac{x^3 + 22x^2 + 14x - 17}{(x-3)(x+2)^3} dx$

Solution:

$$\int \frac{x^3 + 22x^2 + 14x - 17}{(x-3)(x+2)^3} dx$$

Let

$$\frac{x^3 + 22x^2 + 14x - 17}{(x-3)(x+2)^3} = \frac{A}{x-3} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{D}{(x+2)^3} \quad (1)$$

Where A, B, C and D are constant of partial fractions which are to be determined.

Multiply $(x-3)(x+2)^3$ on both sides in (1)

$$x^3 + 22x^2 + 14x - 17 = A(x+2)^3 + B(x-3)(x+2)^2 + C(x-3)(x+2) + D(x-3) \quad (2)$$

$$x^3 + 22x^2 + 14x - 17 = A(x^3 + 8 + 6x^2 + 12x) + B(x-3)(x^2 + 4 + 4x) + C(x^2 - x - 6) + Dx - 3D$$

$$x^3 + 22x^2 + 14x - 17 = Ax^3 + 8A + 6Ax^2 + 12Ax + B(x^3 + 4x + 4x^2 - 3x^2 - 12 - 12x) + Cx^2 - Cx - 6C + Dx - 3D$$

$$x^3 + 22x^2 + 14x - 17 = Ax^3 + 8A + 6Ax^2 + 12Ax + Bx^3 - 8Bx + Bx^2 - 12B + Cx^2 - Cx - 6C + Dx - 3D$$

$$x^3 + 22x^2 + 14x - 17 = (A+B)x^3 + (6A+B+C)x^2 + (12A-8B-C+D)x + (8A-12B-6C-3D) \quad (3)$$

To find A

Put $x - 3 = 0$

$x = 3$ in eq. (2)

$$(3)^3 + 22(3)^2 + 14(3) - 17 = A(3+2)^3$$

$$27 + 22(9) + 42 - 17 = A(5)^3$$

$$27 + 198 + 42 - 17 = 125A$$

$$250 = 125A$$

$$A = \frac{250}{125}$$

$$\boxed{A = 2}$$

To find D

Put $x + 2 = 0$

$x = -2$ in (2)

$$(-2)^3 + 22(-2)^2 + 14(-2) - 17 = D(-2-3)$$

$$-8 + 22(4) - 28 - 17 = D(-5)$$



$$-8 + 88 - 28 - 17 = -5D$$

$$-5D = 35$$

$$D = \frac{35}{-5}$$

$$\boxed{D = -7}$$

To find B and C comparing the coefficients of x^3 , x^2 in (3)

$$A + B = 1$$

$$B = 1 - A$$

$$= 1 - 2$$

$$\boxed{B = -1}$$

$$6A + B + C = 22$$

$$C = 22 - 6A - B$$

$$= 22 - 6(2) - (-1)$$

$$= 22 - 12 + 1$$

$$\boxed{C = 11}$$

∴ From eq. (1)

$$\frac{x^3 + 22x^2 + 14x - 17}{(x-3)(x+2)^3} = \frac{2}{x-3} + \frac{-1}{x+2} + \frac{11}{(x+2)^2} + \frac{-7}{(x+2)^3}$$

Integrate

$$\int \frac{x^3 + 22x^2 + 14x - 17}{(x-3)(x+2)^3} dx = 2 \int \frac{dx}{x-3} - \int \frac{dx}{x+2} + 11 \int \frac{dx}{(x+2)^2} - 7 \int \frac{dx}{(x+2)^3}$$

$$= 2 \ln |x-3| - \ln |x+2| + 11 \int (x+2)^{-2} dx - 7 \int (x+2)^{-3} dx$$

$$= 2 \ln |x-3| - \ln |x+2| + 11 \frac{(x+2)^{-2+1}}{-2+1} - 7 \frac{(x+2)^{-3+1}}{-3+1} + c$$

$$= 2 \ln |x-3| - \ln |x+2| + 11 \frac{(x+2)^{-1}}{-1} - 7 \frac{(x+2)^{-2}}{-2} + c$$

$$= \boxed{2 \ln |x-3| - \ln |x+2| - \frac{11}{x+2} + \frac{7}{2(x+2)^2} + c} \quad \text{Ans.}$$

Q.18 $\int \frac{x-2}{(x+1)(x^2+1)} dx$

Solution:

$$\int \frac{x-2}{(x+1)(x^2+1)} dx$$



Where

$$\frac{x-2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \text{ ——— (1)}$$

Where A, B and C are constant of partial fractions which are to be determined.

Multiplying $(x+1)(x^2+1)$ on both sides in (1)

$$x-2 = A(x^2+1) + (Bx+C)(x+1) \text{ ——— (2)}$$

$$x-2 = Ax^2 + A + Bx^2 + Bx + Cx + C$$

$$x-2 = (A+B)x^2 + (B+C)x + (A+C) \text{ ——— (3)}$$

To find A

$$\text{Put } x+1 = 0$$

$$x = -1 \text{ in eq. (2)}$$

$$-1-2 = A[(-1)^2+1]$$

$$-3 = A(1+1)$$

$$-3 = 2A$$

$$\boxed{A = \frac{-3}{2}}$$

To find B and C comparing the coefficients of x^2 , x in eq. (3)

$$A+B = 0$$

$$B = -A$$

$$B = -\left(\frac{-3}{2}\right)$$

$$\boxed{B = \frac{3}{2}}$$

$$B+C = 1$$

$$C = 1-B$$

$$= 1 - \frac{3}{2}$$

$$= \frac{2-3}{2}$$

$$\boxed{C = \frac{-1}{2}}$$

∴ From eq. (1)

$$\frac{x-2}{(x+1)(x^2+1)} = \frac{\frac{-3}{2}}{x+1} + \frac{\frac{3}{2}x - \frac{1}{2}}{x^2+1}$$



Integrate

$$\int \frac{x-2}{(x+1)(x^2+1)} dx = \frac{-3}{2} \int \frac{dx}{x+1} + \frac{1}{2} \int \frac{3x-1}{x^2+1} dx$$

$$= \frac{-3}{2} \ln|x+1| + \frac{1}{2} \int \frac{3x}{x^2+1} dx - \frac{1}{2} \int \frac{dx}{x^2+1}$$

$$= \frac{-3}{2} \ln|x+1| + \frac{3}{2 \cdot 2} \int \frac{2x}{x^2+1} dx - \frac{1}{2} \tan^{-1} x$$

$$= \boxed{\frac{-3}{2} \ln|x+1| + \frac{3}{4} \ln|x^2+1| - \frac{1}{2} \tan^{-1}(x) + c}$$

Ans.

Q.19 $\int \frac{x}{(x-1)(x^2+1)} dx$

Solution:

$$\int \frac{x}{(x-1)(x^2+1)} dx$$

Let

$$\frac{x}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} \dots\dots\dots (1)$$

where A, B and C are constant of partial fractions, which are to be determined.

Multiplying $(x-1)(x^2+1)$ on both sides in (1)

$$x = A(x^2+1) + (Bx+C)(x-1) \dots\dots\dots (2)$$

$$x = Ax^2 + A + Bx^2 - Bx + Cx - C$$

$$x = (A+B)x^2 + (-B+C)x + (A-C) \dots\dots\dots (3)$$

To find A

$$\text{Put } x-1 = 0$$

$$x = 1 \text{ in eq. (2)}$$

$$1 = A(1+1)$$

$$1 = 2A$$

$$\boxed{A = \frac{1}{2}}$$

To find B and C comparing the coefficients of x^2 and x in eq. (3)

$$A+B = 0$$

$$\frac{1}{2} + B = 0$$

$$\boxed{B = -\frac{1}{2}}$$

$$-B+C = 1$$

$$C = 1+B$$



$$C = 1 - \frac{1}{2}$$

$$C = \frac{2-1}{2}$$

$$\boxed{C = \frac{1}{2}}$$

∴ From equation (1)

$$\frac{x}{(x-1)(x^2+1)} = \frac{\frac{1}{2}}{x-1} + \frac{-\frac{1}{2}x + \frac{1}{2}}{x^2+1}$$

Integrate

$$\begin{aligned} \int \frac{x}{(x-1)(x^2+1)} dx &= \frac{1}{2} \int \frac{dx}{x-1} - \frac{1}{2} \int \frac{x-1}{x^2+1} dx \\ &= \frac{1}{2} \ln |x-1| - \frac{1}{2} \int \frac{2x}{x^2+1} dx + \frac{1}{2} \int \frac{dx}{x^2+1} \\ &= \frac{1}{2} \ln |x-1| - \frac{1}{4} \ln |x^2+1| + \frac{1}{2} \tan^{-1} x + C \quad \text{Ans.} \end{aligned}$$

Q.20 $\int \frac{9x-7}{(x+3)(x^2+1)} dx$

Solution:

$$\int \frac{9x-7}{(x+3)(x^2+1)} dx$$

Let

$$\frac{9x-7}{(x+3)(x^2+1)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+1} \quad \text{..... (1)}$$

where A, B and C are constant of partial fractions, which are to be determined.

Multiplying $(x+3)(x^2+1)$ on both sides in (1)

$$9x-7 = A(x^2+1) + (Bx+C)(x+3) \quad \text{..... (2)}$$

$$9x-7 = Ax^2 + A + Bx^2 + 3Bx + Cx + 3C$$

$$9x-7 = (A+B)x^2 + (3B+C)x + (A+3C) \quad \text{..... (3)}$$

To find A

$$\text{Put } x+3 = 0$$

$$x = -3 \text{ in eq. (2)}$$

$$9(-3)-7 = A[(-3)^2+1]$$

$$-27-7 = A(9+1)$$



$$10A = -34$$

$$A = \frac{-34}{10}$$

$$\boxed{A = -\frac{17}{5}}$$

To find B and C comparing the coefficients of x^2 and x in eq. (3)

$$A + B = 0$$

$$\frac{-17}{5} + B = 0$$

$$\boxed{B = \frac{17}{5}}$$

$$3B + C = 9$$

$$C = 9 - 3B$$

$$= 9 - 3\left(\frac{17}{5}\right)$$

$$= 9 - \frac{51}{5}$$

$$= \frac{45 - 51}{5} = \frac{-6}{5}$$

∴ From equation (1)

$$\frac{9x - 7}{(x + 3)(x^2 + 1)} = \frac{\frac{-17}{5}}{x + 3} + \frac{\frac{17}{5}x + \frac{-6}{5}}{x^2 + 1}$$

Integrate

$$\begin{aligned} \int \frac{9x - 7}{(x + 3)(x^2 + 1)} dx &= \frac{-17}{5} \int \frac{dx}{x + 3} + \frac{1}{5} \int \frac{17x - 6}{x^2 + 1} dx \\ &= \frac{-17}{5} \ln |x + 3| + \frac{17}{5 \cdot 2} \int \frac{2x}{x^2 + 1} dx - \frac{6}{5} \int \frac{dx}{x^2 + 1} \\ &= \frac{-17}{5} \ln |x + 3| + \frac{17}{10} \ln |x^2 + 1| - \frac{6}{5} \tan^{-1} x + C \end{aligned}$$

Q.21 $\int \frac{1 + 4x}{(x - 3)(x^2 + 4)} dx$

Solution:

$$\int \frac{1 + 4x}{(x - 3)(x^2 + 4)} dx$$



Let

$$\frac{1+4x}{(x-3)(x^2+4)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+4} \dots\dots\dots (1)$$

where A, B and C are constant of partial fractions, which are to be determined.

Multiplying $(x-3)(x^2+4)$ on both sides in (1)

$$1+4x = A(x^2+4) + (Bx+C)(x-3) \dots\dots\dots (2)$$

$$1+4x = Ax^2 + 4A + Bx^2 - 3Bx + Cx - 3C$$

$$1+4x = (A+B)x^2 + (-3B+C)x + (4A-3C) \dots\dots\dots (3)$$

To find A

$$\text{Put } x-3 = 0$$

$$x = 3 \text{ in eq. (2)}$$

$$1+4(3) = A[(-3)^2+4]$$

$$1+12 = A(9+4)$$

$$13A = 13$$

$$\boxed{A = 1}$$

To find B and C comparing the coefficients of x^2 and x in eq. (3)

$$A+B = 0$$

$$1+B = 0$$

$$\boxed{B = -1}$$

$$-3B+C = 4$$

$$C = 4+3B$$

$$= 4+3(-1)$$

$$= 4-3$$

$$\boxed{C = 1}$$

$\therefore$ From equation (1)

$$\frac{1+4x}{(x-3)(x^2+4)} = \frac{1}{x-3} + \frac{-x+1}{x^2+4}$$

Integrate

$$\begin{aligned} \int \frac{1+4x}{(x-3)(x^2+4)} dx &= \int \frac{dx}{x-3} - \int \frac{x-1}{x^2+4} dx \\ &= \ln|x-3| - \frac{1}{2} \int \frac{2x}{x^2+4} dx + \int \frac{dx}{x^2+4} \\ &= \ln|x-3| - \frac{1}{2} \ln|x^2+4| + \frac{1}{2} \tan^{-1} \frac{x}{2} + C \end{aligned}$$



Q.22 $\int \frac{12}{x^3 + 8} dx$

Solution:

$$\begin{aligned} & \int \frac{12}{x^3 + 8} dx \\ &= \int \frac{12}{(x)^3 + (2)^3} dx \\ &= \int \frac{12}{(x + 2)(x^2 - 2x + 4)} dx \end{aligned}$$

Let

$$\frac{12}{(x + 2)(x^2 - 2x + 4)} = \frac{A}{x + 2} + \frac{Bx + C}{x^2 - 2x + 4} \quad \text{———— (1)}$$

Where A, B, C and D are constant of partial fractions which are to be determined.

Multiplying $(x + 2)(x^2 - 2x + 4)$ on both sides in (1)

$$12 = A(x^2 - 2x + 4) + (Bx + C)(x + 2) \quad \text{———— (2)}$$

$$12 = Ax^2 - 2Ax + 4A + Bx^2 + 2Bx + Cx + 2C$$

$$12 = (A + B)x^2 + (-2A + 2B + C)x + (4A + 2C) \quad \text{———— (3)}$$

To find A

Put $x + 2 = 0$

$$x = -2 \quad \text{in (2)}$$

$$12 = A[(-2)^2 - 2(-2) + 4]$$

$$12 = A(4 + 4 + 4)$$

$$12 = A(12)$$

$$A = \frac{12}{12}$$

$$\boxed{A = 1}$$

To find B and C comparing the coefficient to of x^2 and x in eq. (3)

$$A + B = 0$$

$$B = -A$$

$$\boxed{B = -1}$$

$$-2A + 2B + C = 0$$

$$C = 2A - 2B$$

$$= 2(1) - 2(-1)$$

$$= 2 + 2$$



$$C = 4$$

∴ From eq. (1)

$$\frac{12}{(x+2)(x^2-2x+4)} = \frac{1}{x+2} + \frac{-x+4}{x^2-2x+4}$$

Integrate

$$\begin{aligned} \int \frac{12}{(x+2)(x^2-2x+4)} dx &= \int \frac{dx}{x+2} - \frac{1}{2} \int \frac{2(x-4)}{x^2-2x+4} dx \\ &= \ln|x+2| - \frac{1}{2} \int \frac{2x-8}{x^2-2x+4} dx \\ &= \ln|x+2| - \frac{1}{2} \int \frac{2x-2-6}{x^2-2x+4} dx \\ &= \ln|x+2| - \frac{1}{2} \int \frac{2x-2}{x^2-2x+4} dx + \frac{6}{2} \int \frac{dx}{x^2-2x+4} \\ &= \ln|x+2| - \frac{1}{2} \ln|x^2-2x+4| + 3 \int \frac{dx}{x^2-2x+1-1+4} \\ &= \ln|x+2| - \frac{1}{2} \ln|x^2-2x+4| + 3 \int \frac{dx}{(x-1)^2+3} \\ &= \ln|x+2| - \frac{1}{2} \ln|x^2-2x+4| + 3 \int \frac{dx}{(x-1)^2+(\sqrt{3})^2} \\ &= \ln|x+2| - \frac{1}{2} \ln|x^2-2x+4| + \frac{3}{\sqrt{3}} \tan^{-1} \left(\frac{x-1}{\sqrt{3}} \right) + c \\ &= \boxed{\ln|x+2| - \frac{1}{2} \ln|x^2-2x+4| + \sqrt{3} \tan^{-1} \left(\frac{x-1}{\sqrt{3}} \right) + c} \quad \text{Ans.} \end{aligned}$$

Q.24 $\int \frac{2x^2+5x+3}{(x-1)^2(x^2+4)} dx$

Solution:

$$\int \frac{2x^2+5x+3}{(x-1)^2(x^2+4)} dx$$

Let

$$\frac{2x^2+5x+3}{(x-1)^2(x^2+4)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+4} \quad \text{———— (1)}$$

Where A, B, C and D are constant of partial fractions which are to be determined.

Multiplying $(x-1)^2(x^2+4)$ on both sides in (1)

$$2x^2+5x+3 = A(x-1)(x^2+4) + B(x^2+4) + (Cx+D)(x-1)^2 \quad \text{———— (2)}$$



$$\begin{aligned}
 2x^2 + 5x + 3 &= A(x^3 + 4x - x^2 - 4) + Bx^2 + 4B + (Cx + D)(x^2 + 1 - 2x) \\
 2x^2 + 5x + 3 &= Ax^3 + 4Ax - Ax^2 - 4A + Bx^2 + 4B + Cx^3 + Cx - 2Cx^2 + Dx^2 + D - 2Dx \\
 2x^2 + 5x + 3 &= (A+C)x^3 + (-A+B-2C+D)x^2 + (4A+C-2D)x + (-4A+4B+D) \quad \text{--- (3)}
 \end{aligned}$$

To find B

$$\text{Put } x - 1 = 0$$

$$x = 1 \text{ in (1)}$$

$$2(1)^2 + 5(1) + 3 = B(1 + 4)$$

$$2 + 5 + 3 = 5B$$

$$10 = 5B$$

$$B = \frac{10}{5}$$

$$\boxed{B = 2}$$

To find A, C and D comparing the coefficients of x^3 , x^2 , x and constant in eq. (3)

$$A + C = 0 \quad \text{--- (i)}$$

$$-A + B - 2C + D = 2 \quad \text{--- (ii)}$$

$$4A + C - 2D = 5 \quad \text{--- (iii)}$$

$$-4D + 4B + D = 3 \quad \text{--- (iv)}$$

Put $B = 2$ in eq (ii) and (iv)

$$-A + 2 - 2C + D = 2, \quad -4A + 4(2) + D = 3$$

$$-A - 2C + D = 0 \quad \text{--- (v)} \quad -4A + 8 + D = 3$$

$$-4A + D = -5 \quad \text{--- (vi)}$$

Adding eq. (i) and eq. (v)

$$A + C = 0$$

$$-A - 2C + D = 0$$

$$-C + D = 0 \quad \text{--- (vii)}$$

$$-C = -D$$

$$C = D$$

Put $C = D$ in eq (i)

$$A + D = 0 \quad \text{--- (viii)}$$

Eq. (vi) + eq. (viii), $\times 4$ we get

$$-4A + D = -5$$

$$4A + 4D = 0$$

$$5D = -5$$

$$\boxed{D = -1}$$



Put $D = -1$ in eq. (vi)

$$-4A - 1 = -5$$

$$-4A = -4$$

$$\boxed{A = 1}$$

Put $A = 1$ in eq. (i)

$$1 + C = 0$$

$$\boxed{C = -1}$$

∴ From eq. (i)

$$\frac{2x^2 + 5x + 3}{(x-1)^2(x^2+4)} = \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{-x-1}{x^2+4}$$

Integrate

$$\begin{aligned} \int \frac{2x^2 + 5x + 3}{(x-1)^2(x^2+4)} dx &= \int \frac{dx}{x-1} + 2 \int \frac{dx}{(x-1)^2} - \int \frac{x+1}{x^2+4} dx \\ &= \ln|x-1| - \frac{2}{x-1} - \frac{1}{2} \int \frac{2x}{x^2+4} dx - \int \frac{dx}{x^2+4} \\ &= \ln|x-1| - \frac{2}{x-1} - \frac{1}{2} \ln|x^2+4| - \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + c \end{aligned}$$

Q.25 $\int \frac{2x^2 - x - 7}{(x+2)^2(x^2+x+1)} dx$

Solution:

$$\int \frac{2x^2 - x - 7}{(x+2)^2(x^2+x+1)} dx$$

Let

$$\frac{2x^2 - x - 7}{(x+2)^2(x^2+x+1)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{Cx+D}{x^2+x+1} \quad \text{———— (1)}$$

Where A, B, C and D are constant of partial fractions which are to be determined.

Multiplying $(x+2)^2(x^2+x+1)$ on both sides in (1)

$$2x^2 - x - 7 = A(x+2)(x^2+x+1) + B(x^2+x+1) + (Cx+D)(x+2)^2 \quad \text{———— (2)}$$

$$2x^2 - x - 7 = A(x^3+x^2+x+2x^2+2x+2) + Bx^2+Bx+B+(Cx+D)(x^2+4x+4)$$

$$2x^2 - x - 7 = Ax^3+3Ax^2+3Ax+2A+Bx^2+Bx+B+Cx^3+4Cx^2+4Cx+Dx^2+4Dx+4D$$

$$2x^2 - x - 7 = (A+C)x^3 + (3A+B+4C+D)x^2 + (3A+B+4C+4D)x + (2A+B+4D) \quad \text{———— (3)}$$

To find B

Put $x+2 = 0$



$x = -2$ in equation (2)

$$2(-2)^2 - (-2) - 7 = B[(-2)2 + (-2) + 1]$$

$$2(4) + 2 - 7 = B(4 - 2 + 1)$$

$$3B = 8 - 5$$

$$B = \frac{3}{3} \Rightarrow \boxed{B = 1}$$

To find A, C and D comparing the coefficients of x^3 , x^2 , x and constant

$$A + C = 0 \quad \text{--- (i)}$$

$$3A + B + 4C + D = 2 \quad \text{--- (ii)}$$

$$3A + B + 4C + 4D = -1 \quad \text{--- (iii)}$$

$$2A + B + 4D = -7 \quad \text{--- (iv)}$$

eq. (ii) - eq. (iii), we get

$$3A + B + 4C + D = 2$$

$$-3A - B - 4C - 4D = -1$$

$$-3D = 3$$

$$D = \frac{3}{-3} \Rightarrow \boxed{D = -1}$$

Put $B = 1$, $D = -1$ in equation (iv)

$$2A + 1 + 4(-1) = -7$$

$$2A + 1 - 4 = -7$$

$$2A - 3 = -7$$

$$2A = -7 + 3$$

$$2A = -4$$

$$A = \frac{-4}{2} \Rightarrow \boxed{A = -2}$$

Put $A = -2$ in equation (i)

$$-2 + C = 0$$

$$\boxed{C = 2}$$

$\therefore$ From equation (1)

$$\frac{2x^2 - x - 7}{(x+2)^2(x^2+x+1)} = \frac{-2}{x+2} + \frac{1}{(x+2)^2} + \frac{2x-1}{x^2+x+1}$$

Integrate

$$\int \frac{2x^2 - x - 7}{(x+2)^2(x^2+x+1)} dx = -2 \int \frac{dx}{x+2} + \int (x+2)^{-2} dx + \int \frac{2x-1}{x^2+x+1} dx$$



$$\begin{aligned}
&= -2 \ln |x+2| + \frac{(x+2)^{-1}}{-1} + \int \frac{2x+1-1}{x^2+x+1} dx \\
&= -2 \ln |x+2| - \frac{1}{x+2} + \int \frac{2x+1}{x^2+x+1} dx - 2 \int \frac{dx}{x^2+x+1} \\
&= -2 \ln |x+2| - \frac{1}{x+2} + \ln |x^2+x+1| - 2 \int \frac{dx}{x^2+x+\frac{1}{4}-\frac{1}{4}+1} \\
&= -2 \ln |x+2| - \frac{1}{x+2} + \ln |x^2+x+1| - 2 \int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}} \\
&= -2 \ln |x+2| - \frac{1}{x+2} + \ln |x^2+x+1| - 2 \int \frac{dx}{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} \\
&= -2 \ln |x+2| - \frac{1}{x+2} + \ln |x^2+x+1| - 2 \frac{1}{\sqrt{3}/2} \tan^{-1} \left(\frac{x+\frac{1}{2}}{\sqrt{3}/2} \right) + c \\
&= -2 \ln |x+2| - \frac{1}{x+2} + \ln |x^2+x+1| - \frac{4}{\sqrt{3}} \tan^{-1} \left(\frac{2x+1}{\sqrt{3}} \right) + c \quad \text{Ans.}
\end{aligned}$$

Q.26 $\int \frac{3x+1}{(4x^2+1)(x^2-x+1)} dx$

Solution:

$$\int \frac{3x+1}{(4x^2+1)(x^2-x+1)} dx$$

Let

$$\frac{3x+1}{(4x^2+1)(x^2-x+1)} = \frac{Ax+B}{4x^2+1} + \frac{Cx+D}{x^2-x+1} \quad \text{———— (1)}$$

Where A, B, C and D are constant of partial fractions which are to be determined.

Multiplying $(4x^2+1)(x^2-x+1)$ on both sides in (1)

$$3x+1 = (Ax+B)(x^2-x+1) + (Cx+D)(4x^2+1)$$

$$3x+1 = Ax^3 - Ax^2 + Ax + Bx^2 - Bx + B + 4Cx^3 + Cx + 4Dx^2 + D$$

$$3x+1 = (A+4C)x^3 + (-A+B+4D)x^2 + (A-B+C)x + (B+D)$$

Comparing the coefficients of x^3 , x^2 , x and constant.

$$A+4C = 0 \quad \text{———— (i)}$$

$$-A+B+4D = 0 \quad \text{———— (ii)}$$

$$A-B+C = 3 \quad \text{———— (iii)}$$



$$B + D = 1 \quad \text{————— (iv)}$$

Eq. (ii) + Eq. (iii), we get

$$-A + B + 4D = 0$$

$$\underline{A - B + C = 3}$$

$$C + 4D = 3 \quad \text{————— (v)}$$

Eq. (iii) + Eq. (iv), we get

$$A - B + C = 3$$

$$\underline{B + D = 1}$$

$$A + C + D = 4 \quad \text{————— (vi)}$$

From equation (i)

$$A + 4C = 0$$

$$A = -4C \quad \text{————— (vii)}$$

Put $A = -4C$ in equation (vi)

$$-4C + C + D = 4$$

$$-3C + D = 4 \quad \text{————— (viii)}$$

Eq. (v) $\times 3$ + Eq. (viii), we get

$$3C + 12D = 9$$

$$\underline{-3C + D = 4}$$

$$13D = 13$$

$$D = \frac{13}{13}$$

$$\boxed{D = 1}$$

Put $D = 1$ in equation (viii)

$$-3C + 1 = 4$$

$$-3C = 4 - 1$$

$$C = \frac{3}{-3}$$

$$\boxed{C = -1}$$

Put $C = -1$ in equation (vii)

$$A = -4(-1)$$

$$\boxed{A = 4}$$

Put $D = 1$ in equation (iv)

$$B + 1 = 1$$

$$B = 1 - 1$$



$$B = 0$$

∴ From equation (1)

$$\frac{3x+1}{(4x^2+1)(x^2-x+1)} = \frac{4x+0}{4x^2+1} + \frac{-x+1}{x^2-x+1}$$

Integrate

$$\begin{aligned} \int \frac{3x+1}{(4x^2+1)(x^2-x+1)} dx &= \frac{1}{2} \int \frac{8x}{4x^2+1} dx - \int \frac{x-1}{x^2-x+1} dx \\ &= \frac{1}{2} \ln |4x^2+1| - \frac{1}{2} \int \frac{2x-2}{x^2-x+1} dx \\ &= \frac{1}{2} \ln |4x^2+1| - \frac{1}{2} \int \frac{2x-1-1}{x^2-x+1} dx \\ &= \frac{1}{2} \ln |4x^2+1| - \frac{1}{2} \int \frac{2x-1}{x^2-x+1} dx + \frac{1}{2} \int \frac{dx}{x^2-x+1} \\ &= \frac{1}{2} \ln |4x^2+1| - \frac{1}{2} \ln |x^2-x+1| + \frac{1}{2} \int \frac{dx}{x^2-x+\frac{1}{4}-\frac{1}{4}+1} \\ &= \frac{1}{2} [\ln |4x^2+1| - \ln |x^2-x+1|] + \frac{1}{2} \int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}} \\ &= \frac{1}{2} \ln \left| \frac{4x^2+1}{x^2-x+1} \right| + \frac{1}{2} \int \frac{dx}{(x-\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2} \\ &= \ln \sqrt{\frac{4x^2+1}{x^2-x+1}} + \frac{1}{2} \cdot \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1} \left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}} \right) + c \\ &= \ln \sqrt{\frac{4x^2+1}{x^2-x+1}} + \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2x-1}{\sqrt{3}} \right) + c \quad \text{Ans.} \end{aligned}$$

Q.27 $\int \frac{4x+1}{(x^2+4)(x^2+4x+5)} dx$

Solution:

$$\int \frac{4x+1}{(x^2+4)(x^2+4x+5)} dx$$



Let

$$\frac{4x+1}{(x^2+4)(x^2+4x+5)} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{x^2+4x+5} \quad \text{————— (1)}$$

Where A, B, C and D are constant of partial fractions which are to be determined.

Multiplying $(x^2+4)(x^2+4x+5)$ on both sides in (1)

$$4x+1 = (Ax+B)(x^2+4x+5) + (Cx+D)(x^2+4)$$

$$4x+1 = Ax^3 + 4Ax^2 + 5Ax + Bx^2 + 4Bx + 5B + Cx^3 + 4Cx + Dx^2 + 4D$$

$$4x+1 = (A+C)x^3 + (4A+B+D)x^2 + (5A+4B+4C)x + (5B+4D)$$

Comparing the coefficients of x^3 , x^2 , x and constant.

$$A+C = 0 \quad \text{————— (i)}$$

$$4A+B+D = 0 \quad \text{————— (ii)}$$

$$5A+4B+4C = 4 \quad \text{————— (iii)}$$

$$5B+4D=1 \quad \text{————— (iv)}$$

Eq. (iii) – Eq. (i) $\times 4$, we get

$$5A+4B+4C = 4$$

$$\underline{-4A \quad \pm \quad 4C = 0}$$

$$A+4B = 4 \quad \text{————— (v)}$$

Eq. (ii) $\times 4$ – Eq. (iv), we get

$$16A+4B+4D = 0$$

$$\underline{-5B \pm 4D = -1}$$

$$\underline{16A - B = -1} \quad \text{————— (vi)}$$

Eq. (v) + Eq. (vi) $\times 4$, we get

$$A+4B = 4$$

$$\underline{64A - 4B = -4}$$

$$65A = 0$$

$$\boxed{A = 0}$$

Put $A = 0$ in equation (vi)

$$16(0) - B = -1$$

$$\boxed{B = 1}$$

Put $A = 0$ in equation (i)

$$0 + C = 0$$

$$\boxed{C = 0}$$



Put $B = 1$ in equation (iv)

$$5 + 4D = 1$$

$$4D = 1 - 5$$

$$D = \frac{-4}{4}$$

$$D = -1$$

∴ From equation (1)

$$\frac{4x + 1}{(x^2 + 4)(x^2 + 4x + 5)} = \frac{0(x) + 1}{x^2 + 4} + \frac{0(x) - 1}{x^2 + 4x + 5}$$

Integrate

$$\begin{aligned} \int \frac{4x + 1}{(x^2 + 4)(x^2 + 4x + 5)} dx &= \int \frac{dx}{(x)^2 + (2)^2} - \int \frac{dx}{x^2 + 4x + 5} \\ &= \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) - \int \frac{dx}{x^2 + 4x + 4 - 4 + 5} \\ &= \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) - \int \frac{dx}{(x + 2)^2 + 1} \\ &= \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) - \int \frac{dx}{(x + 2)^2 + (1)^2} \\ &= \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) - \tan^{-1} \left(\frac{x + 2}{1} \right) + c \\ &= \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) - \tan^{-1}(x + 2) + c \quad \text{Ans.} \end{aligned}$$

Q.28 $\int \frac{6a^2}{(x^2 + a^2)(x^2 + 4a^2)} dx$

Solution:

$$\int \frac{6a^2}{(x^2 + a^2)(x^2 + 4a^2)} dx$$

Let

$$\frac{6a^2}{(x^2 + a^2)(x^2 + 4a^2)} = \frac{Ax + B}{x^2 + a^2} + \frac{Cx + D}{x^2 + 4a^2} \quad \text{————— (1)}$$

Where A, B, C and D are constant of partial fractions which are to be determined.

Multiplying $(x^2 + a^2)(x^2 + 4a^2)$ on both sides in (1)

$$6a^2 = (Ax + B)(x^2 + 4a^2) + (Cx + D)(x^2 + a^2)$$

$$6a^2 = Ax^3 + 4a^2Ax + Bx^2 + 4a^2B + Cx^3 + a^2Cx + Dx^2 + Da^2$$

$$6a^2 = (A + C)x^3 + (B + D)x^2 + (4a^2A + a^2C)x + 4a^2B + Da^2$$



Comparing the coefficients of x^3 , x^2 , x and constant.

$$A + C = 0 \quad \text{———— (i)}$$

$$B + D = 0 \quad \text{———— (ii)}$$

$$4a^2A + a^2C = 0 \quad \text{———— (iii)}$$

$$4a^2B + Da^2 = 6a^2 \quad \text{———— (iv)}$$

Eq. (i) $\times a^2$ – Eq. (iii), we get

$$a^2A + a^2C = 0$$

$$-4a^2A + a^2C = 0$$

$$-3a^2A = 0$$

$$\boxed{A = 0}$$

Put $A = 0$ in equation (i)

$$0 + C = 0$$

$$\boxed{C = 0}$$

Eq. (ii) $\times a^2$ – Eq. (iv), we get

$$Ba^2 + Da^2 = 0$$

$$-4a^2B + Da^2 = -6a^2$$

$$-3a^2B = -5a^2$$

$$B = \frac{-6a^2}{-3a^2}$$

$$\boxed{B = 2}$$

Put $B = 2$ in eq. (2)

$$2 + D = 0$$

$$\boxed{D = -2}$$

$\therefore$ From eq. (1)

$$\frac{6a^2}{(x^2 + a^2)(x^2 + 4a^2)} = \frac{0(x) + 2}{x^2 + a^2} + \frac{0(x) - 2}{x^2 + 4a^2}$$

Integrate

$$\begin{aligned} \int \frac{6a^2}{(x^2 + a^2)(x^2 + 4a^2)} dx &= 2 \int \frac{dx}{x^2 + a^2} - 2 \int \frac{dx}{x^2 + 4a^2} \\ &= 2 \cdot \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) - 2 \cdot \frac{1}{2a} \tan^{-1} \left(\frac{x}{2a} \right) + c \end{aligned}$$



$$= \frac{2}{a} \tan^{-1} \left(\frac{x}{a} \right) - \frac{1}{a} \tan^{-1} \left(\frac{x}{2a} \right) + c \quad \text{Ans.}$$

Q.29 $\int \frac{2x^2 - 2}{x^4 + x^2 + 1} dx$

Solution:

$$\begin{aligned} \int \frac{2x^2 - 2}{x^4 + x^2 + 1} dx \\ x^4 + x^2 + 1 &= (x^2)^2 + 2(x^2)(1) + (1)^2 - x^2 \\ &= (x^2 + 1)^2 - (x)^2 \\ &= (x^2 + 1 + x)(x^2 + 1 - x) \\ &= (x^2 + x + 1)(x^2 - x + 1) \end{aligned}$$

$$= \int \frac{2x^2 - 2}{(x^2 + x + 1)(x^2 - x + 1)} dx$$

Let

$$\frac{2x^2 - 2}{(x^2 + x + 1)(x^2 - x + 1)} = \frac{Ax + B}{x^2 + x + 1} + \frac{Cx + D}{x^2 - x + 1} \quad \text{--- (1)}$$

Where A, B, C and D are constant of partial fractions which are to be determined.

Multiplying $(x^2 + x + 1)(x^2 - x + 1)$ on both sides in (1)

$$2x^2 - 2 = (Ax + B)(x^2 - x + 1) + (Cx + D)(x^2 + x + 1)$$

$$2x^2 - 2 = Ax^3 - Ax^2 + Ax + Bx^2 - Bx + B + Cx^3 + Cx^2 + Cx + Dx^2 + Dx + D$$

$$2x^2 - 2 = (A + C)x^3 + (-A + B + C + D)x^2 + (A - B + C + D)x + (B + D)$$

Comparing the coefficients of x^3 , x^2 , x and constant

$$A + C = 0 \quad \text{--- (i)}$$

$$-A + B + C + D = 2 \quad \text{--- (ii)}$$

$$A - B + C + D = 0 \quad \text{--- (iii)}$$

$$B + D = -2 \quad \text{--- (iv)}$$

Put eq. (i) in eq. (iii)

$$-B + D = 0 \quad \text{--- (v)}$$

Adding eq. (iv) and eq. (v)

$$B + D = -2$$

$$-B + D = 0$$

$$2D = -2$$

$$D = \frac{-2}{2}$$



$$\boxed{D = -1}$$

Put $D = -1$ in eq. (iv)

$$B - 1 = -2$$

$$B = -2 + 1$$

$$\boxed{B = -1}$$

Put $B = -1, D = -1$ in eq. (ii)

$$-A - 1 + C - 1 = 2$$

$$-A + C - 2 = 2$$

$$-A + C = 4 \quad \text{———— (iv)}$$

Adding eq. (i) and eq. (vi)

$$A + C = 2$$

$$\frac{-A + C = 4}{2C = 4}$$

$$2C = 4$$

$$C = \frac{4}{2}$$

$$\boxed{C = 2}$$

Put $C = 2$ in eq. (i)

$$A + 2 = 0$$

$$\boxed{A = -2}$$

∴ From eq. (1)

$$\frac{2x^2 - 2}{(x^2 + x + 1)(x^2 - x + 1)} = \frac{-2x - 1}{x^2 + x + 1} + \frac{2x - 1}{x^2 - x + 1}$$

Integrate

$$\begin{aligned} \int \frac{2x^2 - 2}{(x^2 + x + 1)(x^2 - x + 1)} dx &= - \int \frac{2x + 1}{x^2 + x + 1} dx + \int \frac{2x - 1}{x^2 - x + 1} dx \\ &= - \ln |x^2 + x + 1| + \ln |x^2 - x + 1| + c \\ &= \ln \left| \frac{x^2 - x + 1}{x^2 + x + 1} \right| + c \quad \text{Ans.} \end{aligned}$$

Q.30 $\int \frac{3x - 8}{(x^2 - x + 2)(x^2 + x + 2)} dx$

Solution:

$$\int \frac{3x - 8}{(x^2 - x + 2)(x^2 + x + 2)} dx$$



Let

$$\frac{3x - 8}{(x^2 - x + 2)(x^2 + x + 2)} = \frac{Ax + B}{x^2 - x + 2} + \frac{Cx + D}{x^2 + x + 2} \quad (1)$$

Where A, B, C and D are constant of partial fractions which are to be determined

Multiplying $(x^2 - x + 2)(x^2 + x + 2)$ on both sides in eq. (1)

$$3x - 8 = (Ax + B)(x^2 + x + 2) + (Cx + D)(x^2 - x + 2)$$

$$3x - 8 = Ax^3 + Ax^2 + 2Ax + Bx^2 + Bx + 2B + Cx^3 - Cx^2 + 2Cx + Dx^2 - Dx + 2D$$

$$3x - 8 = (A + C)x^3 + (A + B - C + D)x^2 + (2A + B + 2C - D)x + (2B + 2D)$$

Comparing the coefficients of x^3 , x^2 , x and constant

$$A + C = 0 \quad \text{———— (i)}$$

$$A + B - C + D = 0 \quad \text{———— (ii)}$$

$$2A + B + 2C - D = 3 \quad \text{———— (iii)}$$

$$2B + 2D = -8 \quad \text{———— (iv)}$$

From eq. (iii)

$$2A + B + 2C - D = 3$$

$$2(A + C) + B - D = 3$$

$$2(0) + B - D = 3 \quad \because \text{by eq. (i)}$$

$$B - D = 3 \quad \text{———— (v)}$$

Eq. (iv) + Eq. (v) $\times 2$, we get

$$2B + 3D = -8$$

$$\underline{2B - 2D = 6}$$

$$4B = -2$$

$$B = \frac{-2}{4} = \frac{-1}{2}$$

$$\text{Put } B = \frac{-1}{2} \text{ in eq. (v)}$$

$$\frac{-1}{2} - D = 3$$

$$\frac{-1}{2} - 3 = D$$

$$D = \frac{-1 - 6}{2}$$

$$\boxed{D = \frac{-7}{2}}$$

$$\text{Put } B = \frac{-1}{2}, D = \frac{-7}{2} \text{ in eq. (ii)}$$



$$A - \frac{1}{2} - C - \frac{7}{2} = 0$$

$$A - C = \frac{1}{2} + \frac{7}{2}$$

$$A - C = \frac{1+7}{2}$$

$$A - C = \frac{8}{2}$$

$$A - C = 4 \quad \text{———— (vi)}$$

Adding eq. (i) and eq. (vi)

$$A + C = 0$$

$$\underline{A - C = 4}$$

$$2A = 4$$

$$A = \frac{4}{2}$$

$$\boxed{A = 2}$$

Put $A = 2$ in eq (i)

$$2 + C = 0$$

$$\boxed{C = -2}$$

∴ From eq. (1)

$$\frac{3x-8}{(x^2-x+2)(x^2+x+2)} = \frac{2x-\frac{1}{2}}{x^2-x+2} + \frac{-2x-\frac{7}{2}}{x^2+x+2}$$

Integrate

$$\begin{aligned} \int \frac{3x-8}{(x^2-x+2)(x^2+x+2)} dx &= \int \frac{2x-1+1-\frac{1}{2}}{x^2-x+2} dx - \int \frac{2x+1-1+\frac{7}{2}}{x^2+x+2} dx \\ &= \int \frac{2x-1}{x^2-x+2} dx + \frac{1}{2} \int \frac{dx}{x^2-x+2} - \int \frac{2x+1}{x^2+x+2} dx - \frac{5}{2} \int \frac{dx}{x^2+x+2} \\ &= \ln |x^2-x+2| + \frac{1}{2} \int \frac{dx}{x^2-x+\frac{1}{4}-\frac{1}{4}+2} - \ln |x^2+x+2| - \frac{5}{2} \int \frac{dx}{x^2+x+\frac{1}{4}-\frac{1}{4}+2} \\ &= \ln |x^2-x+2| + \frac{1}{2} \int \frac{dx}{(x-\frac{1}{2})^2+\frac{7}{4}} - \ln |x^2+x+2| - \frac{5}{2} \int \frac{dx}{(x+\frac{1}{2})^2+\frac{7}{4}} \end{aligned}$$



$$\begin{aligned}
&= \ln |x^2 + x + 2| + \frac{1}{2} \int \frac{dx}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2} - \ln |x^2 + x + 2| - \frac{5}{2} \int \frac{dx}{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2} \\
&= \ln |x^2 - x + 2| + \frac{1}{2} \cdot \frac{1}{\frac{\sqrt{7}}{2}} \tan^{-1} \left(\frac{x - \frac{1}{2}}{\frac{\sqrt{7}}{2}} \right) - \ln |x^2 + x + 2| - \frac{5}{2} \cdot \frac{1}{\frac{\sqrt{7}}{2}} \tan^{-1} \left(\frac{x + \frac{1}{2}}{\frac{\sqrt{7}}{2}} \right) + c \\
&= \ln \left| \frac{x^2 - x + 2}{x^2 + x + 2} \right| + \frac{1}{\sqrt{7}} \tan^{-1} \left(\frac{2x - 1}{\sqrt{7}} \right) - \frac{5}{\sqrt{7}} \tan^{-1} \left(\frac{2x + 1}{\sqrt{7}} \right) + c \\
&= \ln \left| \frac{x^2 - x + 2}{x^2 + x + 2} \right| + \frac{1}{\sqrt{7}} \left[\tan^{-1} \left(\frac{2x - 1}{\sqrt{7}} \right) - 5 \tan^{-1} \left(\frac{2x + 1}{\sqrt{7}} \right) \right] + c \quad \text{Ans.}
\end{aligned}$$

Q.31 $\int \frac{3x^3 + 4x^2 + 9x + 5}{(x^2 + x + 1)(x^2 + 2x + 3)} dx$

Solution:

$$\int \frac{3x^3 + 4x^2 + 9x + 5}{(x^2 + x + 1)(x^2 + 2x + 3)} dx$$

Let

$$\frac{3x^3 + 4x^2 + 9x + 5}{(x^2 + x + 1)(x^2 + 2x + 3)} = \frac{Ax + B}{x^2 + x + 1} + \frac{Cx + D}{x^2 + 2x + 3} \quad (1)$$

Where A, B, C and D are constant of partial fractions which are to be determined.

Multiplying $(x^2 + x + 1)(x^2 + 2x + 3)$ on both sides in eq. (1)

$$3x^3 + 4x^2 + 9x + 5 = (Ax + B)(x^2 + 2x + 3) + (Cx + D)(x^2 + x + 1)$$

$$3x^3 + 4x^2 + 9x + 5 = Ax^3 + 2Ax^2 + 3Ax + Bx^2 + 2Bx + 3B + Cx^3 + Cx^2 + Cx + Dx^2 + Dx + D$$

$$3x^3 + 4x^2 + 9x + 5 = (A + C)x^3 + (2A + B + C + D)x^2 + (2A + 2B + C + D)x + (3B + D)$$

Comparing the coefficients of x^3 , x^2 , x and constant.

$$A + C = 3 \quad \text{———— (i)}$$

$$2A + B + C + D = 4 \quad \text{———— (ii)}$$

$$3A + 2B + C + D = 9 \quad \text{———— (iii)}$$

$$3B + D = 5 \quad \text{———— (iv)}$$

From eq. (i)

$$A + C = 3$$

$$C = 3 - A \quad \text{———— (v)}$$

From eq. (iv)

$$3B + D = 5$$

$$D = 5 - 3B \quad \text{———— (vi)}$$



Put eq. (v) and eq. (vi) in eq. (ii)

$$2A + B + 3 - A + 5 - 3B = 4$$

$$A - 2B + 8 = 4$$

$$A - 2B = -4 \quad \text{———— (vii)}$$

Put eq. (v) and eq. (vi) in eq. (iii)

$$3A + 2B + 3 - A + 5 - 3B = 9$$

$$2A - B + 8 = 9$$

$$2A - B = 1 \quad \text{———— (viii)}$$

Eq. (vii) – Eq. (viii) $\times 2$, we get

$$A - 3B = -4$$

$$\underline{-4A - 2B = -2}$$

$$-3A = -6$$

$$A = \frac{-6}{-3}$$

$$\boxed{A = 2}$$

Put $A = 2$ in eq. (vii)

$$2 - 2B = -4$$

$$-2B = -4 - 2$$

$$B = \frac{-6}{-2}$$

$$\boxed{B = 3}$$

Put $A = 2$ in eq. (i)

$$2 + C = 3$$

$$C = 3 - 2$$

$$\boxed{C = 1}$$

Put $B = 3$ in eq. (iv)

$$3(3) + D = 5$$

$$9 + D = 5$$

$$D = 5 - 9$$

$$\boxed{D = -4}$$

$\therefore$ From eq. (1)

$$\frac{3x^3 + 4x^2 + 9x + 5}{(x^2 + x + 1)(x^2 + 2x + 3)} = \frac{2x + 3}{x^2 + x + 1} + \frac{x - 4}{x^2 + 2x + 3}$$



Integrate

$$\begin{aligned}
 \int \frac{3x^3 + 4x^2 + 9x + 5}{(x^2 + x + 1)(x^2 + 2x + 3)} dx &= \int \frac{2x + 1 + 2}{x^2 + x + 1} dx + \frac{1}{2} \int \frac{2x - 8 + 2 - 2}{x^2 + 2x + 3} dx \\
 &= \int \frac{2x + 1}{x^2 + x + 1} dx + 2 \int \frac{dx}{x^2 + x + 1} + \frac{1}{2} \int \frac{2x + 2}{x^2 + 2x + 3} dx - \frac{10}{2} \int \frac{dx}{x^2 + 2x + 3} \\
 &= \ln |x^2 + x + 1| + 2 \int \frac{dx}{x^2 + x + \frac{1}{4} - \frac{1}{4} + 1} + \frac{1}{2} \ln |x^2 + 2x + 3| - 5 \int \frac{dx}{x^2 + 2x + 1 - 1 + 3} \\
 &= \ln |x^2 + x + 1| + 2 \int \frac{dx}{(x + \frac{1}{2})^2 + \frac{3}{4}} + \ln |x^2 + 2x + 3|^{\frac{1}{2}} - 5 \int \frac{dx}{(x + 1)^2 + 2} \\
 &= \ln |(x^2 + x + 1) \sqrt{x^2 + 2x + 3}| + 2 \int \frac{dx}{(x + \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} - 5 \int \frac{dx}{(x + 1)^2 + (\sqrt{2})^2} \\
 &= \ln |(x^2 + x + 1) \sqrt{x^2 + 2x + 3}| + 2 \cdot \frac{1}{\sqrt{3}/2} \tan^{-1} \left(\frac{x + \frac{1}{2}}{\sqrt{3}/2} \right) - 5 \cdot \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x + 1}{\sqrt{2}} \right) + c \\
 &= \ln |(x^2 + x + 1) \sqrt{x^2 + 2x + 3}| + \frac{4}{\sqrt{3}} \tan^{-1} \left(\frac{2x + 1}{\sqrt{3}} \right) - \frac{5}{\sqrt{2}} \tan^{-1} \left(\frac{x + 1}{\sqrt{2}} \right) + c
 \end{aligned}$$



EXERCISE 3.6

Q.1 $\int_1^2 (x^2 + 1) dx$

Solution:

$$\begin{aligned} & \int_1^2 (x^2 + 1) dx \\ &= \int_1^2 x^2 dx + \int_1^2 dx \\ &= \left[\frac{x^3}{3} \right]_1^2 + [x]_1^2 \end{aligned}$$

$$\begin{aligned} \therefore \int_a^b f(x) dx &= [\varphi(x)]_a^b \\ &= \varphi(b) - \varphi(a) \end{aligned}$$



$$\begin{aligned}
 &= \frac{1}{3} (2^3 - 1^3) + (2 - 1) \\
 &= \frac{1}{3} (8 - 1) + 1 = \frac{7}{3} + 1 \\
 &= \frac{7+3}{3} = \frac{10}{3} \quad \text{Ans.}
 \end{aligned}$$

Q.2 $\int_{-1}^1 (x^{1/3} + 1) dx$

Solution:

$$\begin{aligned}
 &\int_{-1}^1 (x^{1/3} + 1) dx \\
 &= \int_{-1}^1 x^{1/3} dx + \int_{-1}^1 dx \\
 &= \left[\frac{x^{2/3}}{\frac{2}{3}} \right]_{-1}^1 + [x]_{-1}^1 \\
 &= \frac{3}{2} [(1)^{2/3} - (-1)^{2/3}] + [1 - (-1)] \\
 &= \frac{3}{2} (1 - 1) + 2 \\
 &= \frac{3}{2} (0) + 2 = 2 \quad \text{Ans.}
 \end{aligned}$$

Q.3 $\int_{-2}^0 \frac{1}{(2x-1)^2} dx$

Solution:

$$\begin{aligned}
 &\int_{-2}^0 \frac{1}{(2x-1)^2} dx \\
 &= \frac{1}{2} \int_{-2}^0 (2x-1)^{-2} \cdot 2 dx
 \end{aligned}$$



$$\begin{aligned}
&= \frac{1}{2} \left[\frac{(2x-1)^{-2+1}}{-2+1} \right]_{-2}^0 \\
&= \frac{1}{2} \left[\frac{(2x-1)^{-1}}{-1} \right]_{-2}^0 \\
&= \frac{-1}{2} \left[\frac{1}{2x-1} \right]_{-2}^0 \\
&= \frac{-1}{2} \left[\frac{1}{2(0)-1} - \frac{1}{2(-2)-1} \right] \\
&= \frac{-1}{2} \left[\frac{1}{0-1} - \frac{1}{-4-1} \right] \\
&= \frac{-1}{2} \left(-1 + \frac{1}{5} \right) \\
&= \frac{-1}{2} \left(\frac{-5+1}{5} \right) \\
&= \frac{-1(-4)}{10} \\
&= \frac{2}{5} \quad \text{Ans.}
\end{aligned}$$

Q.4

$$\int_{-6}^2 \sqrt{3-x} \, dx$$

Solution:

$$\begin{aligned}
&\int_{-6}^2 \sqrt{3-x} \, dx \\
&= -\int_{-6}^2 (3-x)^{1/2} \, dx \\
&= -\left[\frac{(3-x)^{2/3}}{\frac{2}{3}} \right]_{-6}^2 \\
&= \frac{-2}{3} [(3-2)^{2/3} - (3+6)^{2/3}] \\
&= \frac{-2}{3} [(1)^{2/3} - (9)^{2/3}]
\end{aligned}$$



$$= \frac{-2}{3} [1 - (3^2)^{2/3}]$$

$$= \frac{-2}{3} (1 - 27)$$

$$= \frac{-2}{3} (-26)$$

$$= \frac{52}{3} \quad \text{Ans.}$$

Q.5 $\int_1^{\sqrt{5}} \sqrt{(2t-1)^3} \, dt$

Solution:

$$A = \int_1^{\sqrt{5}} \sqrt{(2t-1)^3} \, dt$$

$$= \frac{1}{2} \int_1^{\sqrt{5}} (2t-1)^3 \cdot 2 \, dt$$

$$= \frac{1}{2} \left[\frac{(2t-1)^{5/2}}{\frac{5}{2}} \right]_1^{\sqrt{5}}$$

$$= \frac{1}{5} [(2\sqrt{5}-1)^{5/2} - (2-1)^{5/2}]$$

$$= \frac{1}{5} [(2\sqrt{5}-1)^{5/2} - 1] \quad \text{Ans.}$$

Q.6 $\int_2^{\sqrt{5}} x\sqrt{x^2-1} \, dx$

Solution:

$$\int_2^{\sqrt{5}} x\sqrt{x^2-1} \, dx$$

$$= \frac{1}{2} \int_2^{\sqrt{5}} (x^2-1)^{1/2} \cdot 2x \, dx$$

$$= \frac{1}{2} \left[\frac{(x^2-1)^{3/2}}{3/2} \right]_2^{\sqrt{5}}$$



$$= \frac{1}{3} [(5-1)^{3/2} - (4-1)^{3/2}]$$

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$$= \frac{1}{3} [4^{3/2} - 3^{3/2}] = \frac{1}{3} [(2^2)^{3/2} - 3\sqrt{3}]$$

$$= \frac{1}{3} (8 - 3\sqrt{3}) \quad \text{Ans.}$$

Q.7 $\int_1^2 \frac{x}{x^2+2} dx$ (Lhr. Board 2011, Guj. Board 2008)

Solution:

$$\int_1^2 \frac{x}{x^2+2} dx$$

$$= \frac{1}{2} \int_1^2 \frac{2x}{x^2+2} dx$$

$$= \frac{1}{2} [\ln(x^2+2)]_1^2$$

$$= \frac{1}{2} \ln(4+2) - \ln(1+2)$$

$$= \frac{1}{2} (\ln 6 - \ln 3)$$

$$= \frac{1}{2} \ln \frac{6}{3}$$

$$= \frac{1}{2} \ln 2 \quad \text{Ans.}$$

Q.8 $\int_2^3 (x - \frac{1}{x})^2 dx$

Solution:

$$\int_2^3 (x - \frac{1}{x})^2 dx$$

$$= \int_2^3 (x^2 - 2 + \frac{1}{x^2}) dx$$

$$= \int_2^3 x^2 dx - 2 \int_2^3 dx + \int_2^3 x^{-2} dx$$

$$= \left[\frac{x^3}{3} \right]_2^3 - 2 [x]_2^3 + \left[\frac{x-1}{-1} \right]_2^3$$



$$\begin{aligned}
 &= \frac{1}{3} (3^3 - 2^3) - 2(3 - 2) - \left[\frac{1}{x} \right]_2^3 \\
 &= \frac{1}{3} (27 - 8) - 2(1) - \left(\frac{1}{3} - \frac{1}{2} \right) \\
 &= \frac{19}{3} - 2 - \frac{1}{3} + \frac{1}{2} \\
 &= \frac{38 - 12 - 2 + 3}{6} \\
 &= \frac{27}{6} = \frac{9}{2} \quad \text{Ans.}
 \end{aligned}$$

Q.9 $\int_{-1}^1 \left(x + \frac{1}{2}\right) \sqrt{x^2 + x + 1} \, dx$ (Guj. Board 2007)

Solution:

$$\begin{aligned}
 &\int_{-1}^1 \left(x + \frac{1}{2}\right) \sqrt{x^2 + x + 1} \, dx \\
 &= \frac{1}{2} \int_{-1}^1 (x^2 + x + 1)^{1/2} \cdot 2 \left(x + \frac{1}{2}\right) \, dx \\
 &= \frac{1}{2} \int_{-1}^1 (x^2 + x + 1)^{1/2} \cdot (2x + 1) \, dx \\
 &= \frac{1}{2} \left[\frac{(x^2 + x + 1)^{3/2}}{\frac{3}{2}} \right]_{-1}^1 \\
 &= \frac{1}{3} \left[(1 + 1 + 1)^{3/2} - (1 - 1 + 1)^{3/2} \right] \\
 &= \frac{1}{3} [3^{3/2} - 1^{3/2}] \\
 &= \frac{1}{3} (3\sqrt{3} - 1) \\
 &= \frac{3\sqrt{3}}{3} - \frac{1}{3} \\
 &= \sqrt{3} - \frac{1}{3} \quad \text{Ans.}
 \end{aligned}$$



Q.10 $\int_0^3 \frac{dx}{x^2 + 9}$

Solution:

$$\begin{aligned} & \int_0^3 \frac{dx}{x^2 + 9} \\ &= \int_0^3 \frac{dx}{(x)^2 + (3)^2} \\ &= \left[\frac{1}{3} \tan^{-1} \left(\frac{x}{3} \right) \right]_0^3 \\ &= \frac{1}{3} \left[\tan^{-1} \left(\frac{3}{3} \right) - \tan^{-1} \left(\frac{0}{3} \right) \right] \\ &= \frac{1}{3} [\tan^{-1}(1) - \tan^{-1}(0)] \\ &= \frac{1}{3} \left(\frac{\pi}{4} - 0 \right) \\ &= \frac{\pi}{12} \quad \text{Ans.} \end{aligned}$$

Q.11 $\int_{\pi/6}^{\pi/3} \cos t \, dt$

Solution:

$$\begin{aligned} & \int_{\pi/6}^{\pi/3} \cos t \, dt = [\sin t]_{\pi/6}^{\pi/3} \\ &= \sin \frac{\pi}{3} - \sin \frac{\pi}{6} \\ &= \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3} - 1}{2} \quad \text{Ans.} \end{aligned}$$

Q.12 $\int_2^1 \left(x + \frac{1}{x}\right)^{1/2} \left(1 - \frac{1}{x^2}\right) dx$ (Lhr. Board 2011)

Solution:

$$\int_2^1 \left(x + \frac{1}{x}\right)^{1/2} \left(1 - \frac{1}{x^2}\right) dx$$



$$\begin{aligned}
 &= \left[\frac{\left(x + \frac{1}{x}\right)^{3/2}}{\frac{3}{2}} \right]_1^2 \\
 &= \frac{2}{3} \left[\left(2 + \frac{1}{2}\right)^{3/2} - \left(1 + \frac{1}{1}\right)^{3/2} \right] \\
 &= \frac{2}{3} \left[\left(\frac{4+1}{2}\right)^{3/2} - 2^{3/2} \right] \\
 &= \frac{2}{3} \left[\left(\frac{5}{2}\right)^{3/2} - 2\sqrt{2} \right] \\
 &= \frac{2}{3} \left[\frac{5\sqrt{5}-8}{2\sqrt{2}} - 2\sqrt{2} \right] = \frac{2}{3} \left[\frac{5\sqrt{5}-8}{2\sqrt{2}} \right] \\
 &= \frac{1}{3\sqrt{2}} (5\sqrt{5}-8) \quad \text{Ans.}
 \end{aligned}$$

Q.13 $\int_1^2 \ln x \, dx$

Solution:

$$\begin{aligned}
 &\int_1^2 \ln x \, dx \\
 &= \int_1^2 1 \cdot \ln x \, dx \\
 &= [\ln x \cdot x]_1^2 - \int_1^2 x \cdot \frac{1}{x} \, dx \\
 &= [2 \ln 2 - 1 \ln 1] - \int_1^2 dx \\
 &= (2 \ln 2 - 0) - [x]_1^2 \\
 &= 2 \ln 2 - (2 - 1) \\
 &= 2 \ln 2 - 1 \quad \text{Ans.}
 \end{aligned}$$

Q.14 $\int_0^2 (e^{x/2} - e^{-x/2}) \, dx$

Solution:

$$\int_0^2 (e^{x/2} - e^{-x/2}) \, dx$$



$$\begin{aligned}
&= \int_0^2 e^{x/2} dx - \int_0^2 e^{-x/2} dx \\
&= \left[\frac{e^{x/2}}{\frac{1}{2}} \right]_0^2 - \left[\frac{e^{-x/2}}{\frac{-1}{2}} \right]_0^2 \\
&= 2(e^{2/2} - e^{0/2}) + 2(e^{-2/2} - e^{-0/2}) \\
&= 2(e - 1 + e^{-1} - 1) \\
&= 2\left(e + \frac{1}{e} - 2\right) \\
&= 2\left(\frac{e^2 + 1 - 2e}{e}\right) \\
&= \frac{2}{e} (e - 1)^2 \quad \text{Ans.}
\end{aligned}$$

Q.15 $\int_0^{\pi/4} \frac{\cos\theta + \sin\theta}{\cos 2\theta + 1} d\theta$

Solution:

$$\begin{aligned}
&\int_0^{\pi/4} \frac{\cos\theta + \sin\theta}{\cos 2\theta + 1} d\theta \\
&= \int_0^{\pi/4} \frac{\cos\theta + \sin\theta}{2\cos^2\theta} \quad \left(\begin{array}{l} \because \cos 2\theta = 2\cos^2\theta - 1 \\ 1 + \cos 2\theta = 2\cos^2\theta \end{array} \right) \\
&= \frac{1}{2} \int_0^{\pi/4} \left(\frac{\cos\theta}{\cos^2\theta} + \frac{\sin\theta}{\cos^2\theta} \right) d\theta \\
&= \frac{1}{2} \int_0^{\pi/4} \left(\frac{1}{\cos\theta} + \frac{1}{\cos\theta} \cdot \frac{\sin\theta}{\cos\theta} \right) d\theta \\
&= \frac{1}{2} \left[\int_0^{\pi/4} \sec\theta d\theta + \int_0^{\pi/4} \sec\theta \tan\theta d\theta \right] \\
&= \frac{1}{2} \left[[\ln |\sec\theta + \tan\theta|]_0^{\pi/4} + [\sec\theta]_0^{\pi/4} \right] \\
&= \frac{1}{2} \left[\ln \left| \sec \frac{\pi}{4} + \tan \frac{\pi}{4} \right| - \ln |\sec 0 + \tan 0| + \sec \frac{\pi}{4} - \sec 0 \right] \\
&= \frac{1}{2} [\ln (\sqrt{2} + 1) - \ln (1 + 0) + \sqrt{2} - 1] \\
&= \frac{1}{2} [\ln (\sqrt{2} + 1) + \sqrt{2} - 1] \quad \text{Ans.}
\end{aligned}$$



Q.16 $\int_0^{\frac{\pi}{6}} \cos^3 \theta \, d\theta$

Solution:

$$\begin{aligned}
 & \int_0^{\frac{\pi}{6}} \cos^3 \theta \, d\theta \\
 &= \int_0^{\frac{\pi}{6}} \cos \theta \cdot \cos^2 \theta \, d\theta \\
 &= \int_0^{\frac{\pi}{6}} \cos \theta (1 - \sin^2 \theta) \, d\theta \\
 &= \int_0^{\frac{\pi}{6}} \cos \theta \, d\theta - \int_0^{\frac{\pi}{6}} \sin^2 \theta \cos \theta \, d\theta \\
 &= [\sin \theta]_0^{\pi/6} - \left[\frac{\sin^3 \theta}{3} \right]_0^{\pi/6} \\
 &= \sin \frac{\pi}{6} - \sin 0 - \frac{1}{3} \left(\sin^3 \frac{\pi}{6} - \sin^3 0 \right) \\
 &= \frac{1}{2} - 0 - \frac{1}{3} \left[\left(\frac{1}{2} \right)^3 - 0 \right] \\
 &= \frac{1}{2} - \frac{1}{3} \left(\frac{1}{8} \right) = \frac{1}{2} - \frac{1}{24} \\
 &= \frac{12-1}{24} = \frac{11}{24} \quad \text{Ans.}
 \end{aligned}$$

Q.17 $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos^2 \theta \cot^2 \theta \, d\theta$

Solution:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos^2 \theta \cot^2 \theta \, d\theta$$



$$\begin{aligned}
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos\theta (\operatorname{cosec}^2\theta - 1) d\theta \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\cos^2\theta \operatorname{cosec}^2\theta - \cos^2\theta) d\theta \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\frac{\cos^2\theta}{\sin^2\theta} - \frac{1 + \cos 2\theta}{2} \right) d\theta \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\cot^2\theta - \frac{1 + \cos 2\theta}{2} \right) d\theta \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\operatorname{cosec}^2\theta - 1 - \frac{1 + \cos 2\theta}{2} \right) d\theta \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\frac{2\operatorname{cosec}^2\theta - 2 - 1 - \cos 2\theta}{2} \right) d\theta \\
&= \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (2\operatorname{cosec}^2\theta - 3 - \cos 2\theta) d\theta \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec}^2\theta d\theta - \frac{3}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} d\theta - \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos 2\theta d\theta \\
&= [-\cot\theta]_{\frac{\pi}{6}}^{\frac{\pi}{4}} - \frac{3}{2} [\theta]_{\frac{\pi}{6}}^{\frac{\pi}{4}} - \frac{1}{2} \left[\frac{\sin 2\theta}{2} \right]_{\frac{\pi}{6}}^{\frac{\pi}{4}} \\
&= -\left[\cot \frac{\pi}{4} - \cot \frac{\pi}{6} \right] - \frac{3}{2} \left(\frac{\pi}{4} - \frac{\pi}{6} \right) - \frac{1}{4} \left[\sin 2 \left(\frac{\pi}{4} \right) - \sin 2 \left(\frac{\pi}{6} \right) \right] \\
&= -1 + \sqrt{3} - \frac{3}{2} \left(\frac{3\pi - 2\pi}{12} \right) - \frac{1}{4} \left(1 - \frac{\sqrt{3}}{2} \right) \\
&= -1 + \sqrt{3} - \frac{\pi}{8} - \frac{1}{4} + \frac{\sqrt{3}}{8}
\end{aligned}$$



$$= \frac{-8 + 8\sqrt{3} - \pi - 2 + \sqrt{3}}{8}$$

$$= \frac{-10 - \pi + 9\sqrt{3}}{8} \quad \text{Ans.}$$

Q.18 $\int_0^{\frac{\pi}{4}} \cos^4 t \, dt$

Solution:

$$\int_0^{\frac{\pi}{4}} \cos^4 t \, dt$$

$$= \int_0^{\frac{\pi}{4}} (\cos^2 t)^2 \, dt$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1 + \cos 2t}{2} \right)^2 \, dt$$

$$= \frac{1}{4} \int_0^{\frac{\pi}{4}} (1 + \cos^2 2t + 2\cos 2t) \, dt$$

$$= \frac{1}{4} \int_0^{\frac{\pi}{4}} \left(1 + \frac{1 + \cos 4t}{2} + 2\cos 2t \right) \, dt$$

$$= \frac{1}{4} \int_0^{\frac{\pi}{4}} \left(\frac{2 + 1 + \cos 4t + 4\cos 2t}{2} \right) \, dt$$

$$= \frac{1}{8} \int_0^{\frac{\pi}{4}} (3 + \cos 4t + 4\cos 2t) \, dt$$

$$= \frac{1}{8} \left[3 \int_0^{\frac{\pi}{4}} dt + \int_0^{\frac{\pi}{4}} \cos 4t \, dt + 4 \int_0^{\frac{\pi}{4}} \cos 2t \, dt \right]$$

$$= \frac{1}{8} \left[3 [t]_0^{\pi/4} + \left[\frac{\sin 4t}{4} \right]_0^{\pi/4} + 4 \left[\frac{\sin 2t}{2} \right]_0^{\pi/4} \right]$$

$$\begin{aligned} \because \cos 2t &= 2 \cos^2 t - 1 \\ 2 \cos^2 t &= 1 + \cos 2t \\ \cos^2 t &= \frac{1 + \cos 2t}{2} \end{aligned}$$

$$\begin{aligned} \therefore \cos^2 t &= \frac{1 + \cos 2t}{2} \\ \cos^2 2t &= \frac{1 + \cos 4t}{2} \end{aligned}$$



$$\begin{aligned}
&= \frac{1}{8} \left[3 \left(\frac{\pi}{4} - 0 \right) + \frac{1}{4} \left(\sin 4 \cdot \frac{\pi}{4} - \sin 0 \right) + 2 \left(\sin 2 \cdot \frac{\pi}{4} - \sin 0 \right) \right] \\
&= \frac{1}{8} \left[\frac{3\pi}{4} + \frac{1}{4} (0 - 0) + 2 (1 - 0) \right] \\
&= \frac{1}{8} \left(\frac{3\pi}{4} + 2 \right) \\
&= \frac{1}{8} \left(\frac{3\pi + 8}{4} \right) \\
&= \frac{3\pi + 8}{32} \quad \text{Ans.}
\end{aligned}$$

Q.19 $\int_0^{\frac{\pi}{3}} \cos^2 \theta \sin \theta \, d\theta$

Solution:

$$\begin{aligned}
&\int_0^{\frac{\pi}{3}} \cos^2 \theta \sin \theta \, d\theta \\
&= - \int_0^{\frac{\pi}{3}} \cos^2 \theta (-\sin \theta) \, d\theta \\
&= - \left[\frac{\cos^3 \theta}{3} \right]_0^{\pi/3} \\
&= \frac{-1}{3} \left[\cos^3 \frac{\pi}{3} - \cos^3 0 \right] \\
&= \frac{-1}{3} \left[\left(\frac{1}{2} \right)^3 - 1 \right] \\
&= \frac{-1}{3} \left(\frac{1}{8} - 1 \right) \\
&= \frac{-1}{3} \left(\frac{1-8}{8} \right) \\
&= \frac{-1}{3} \left(\frac{-7}{8} \right) \\
&= \frac{7}{24} \quad \text{Ans.}
\end{aligned}$$



Q.20 $\int_0^{\frac{\pi}{4}} (1 + \cos^2\theta) \tan^2\theta d\theta$

Solution:

$$\begin{aligned}
 & \int_0^{\frac{\pi}{4}} (1 + \cos^2\theta) \tan^2\theta d\theta \\
 &= \int_0^{\frac{\pi}{4}} (\tan^2\theta + \tan^2\theta \cos^2\theta) d\theta \\
 &= \int_0^{\frac{\pi}{4}} \left(\sec^2\theta - 1 + \frac{\sin^2\theta}{\cos^2\theta} \cdot \cos^2\theta \right) d\theta \\
 &= \int_0^{\frac{\pi}{4}} (\sec^2\theta - 1 + \sin^2\theta) d\theta \\
 &= \int_0^{\frac{\pi}{4}} \left(\sec^2\theta - 1 + \frac{1 - \cos 2\theta}{2} \right) d\theta \\
 &= \int_0^{\frac{\pi}{4}} \left(\frac{2\sec^2\theta - 2 + 1 - \cos 2\theta}{2} \right) d\theta \\
 &= \frac{1}{2} \int_0^{\frac{\pi}{4}} (2 \sec^2\theta - 1 - \cos 2\theta) d\theta \\
 &= \int_0^{\frac{\pi}{4}} \sec^2\theta d\theta - \frac{1}{2} \int_0^{\frac{\pi}{4}} d\theta - \frac{1}{2} \int_0^{\frac{\pi}{4}} \cos 2\theta d\theta \\
 &= \left[\tan\theta \right]_0^{\pi/4} - \frac{1}{2} [\theta] - \frac{1}{2} \left[\frac{\sin 2\theta}{2} \right]_0^{\pi/4} \\
 &= \left(\tan \frac{\pi}{4} - \tan 0 \right) - \frac{1}{2} \left(\frac{\pi}{4} - 0 \right) - \frac{1}{4} \left[\sin 2 \left(\frac{\pi}{4} \right) - \sin 2 (0) \right]
 \end{aligned}$$



$$= 1 - 0 - \frac{\pi}{8} - \frac{1}{4}(1 - 0) = 1 - \frac{\pi}{8} - \frac{1}{4}$$

$$= \frac{8 - \pi - 2}{8} = \frac{6 - \pi}{8} \quad \text{Ans.}$$

Q.21 $\int_0^{\frac{\pi}{4}} \frac{\sec\theta}{\sin\theta + \cos\theta} d\theta$ (Guj. Board 2008)

Solution:

$$\int_0^{\frac{\pi}{4}} \frac{\sec\theta}{\sin\theta + \cos\theta} d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1/\cos^2\theta}{\tan\theta + 1} d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\tan\theta + 1} d\theta$$

$$= [\ln |\tan\theta + 1|]_0^{\pi/4} = \ln |\tan \frac{\pi}{4} + 1| - \ln |\tan 0 + 1|$$

$$= \ln |1 + 1| - \ln |0 + 1| = \ln 2 - 0 = \ln 2 \quad \text{Ans.}$$

Q.22 $\int_{-1}^5 |x - 3| dx$ (Lhr. Board 2009 (S))

Solution:

We know that

$$|x - 3| = \begin{cases} +(x - 3) & \text{If } x > 3 \\ -(x - 3) & \text{If } x < 3 \\ 0 & \text{If } x = 3 \end{cases}$$

$$= \int_{-1}^3 |x - 3| dx + \int_3^5 (x - 3) dx$$

$$= -\int_{-1}^3 (x - 3) dx + \int_3^5 (x - 3) dx$$

$$= -\left[\frac{(x - 3)^2}{2}\right]_{-1}^3 + \left[\frac{(x - 3)^2}{2}\right]_3^5$$



$$= -\left[0 - \frac{16}{2}\right] + \left[\frac{4}{2} - 0\right] = \frac{16}{2} + \frac{4}{2} = 8 + 2 = 10$$

Q.23 $\int_{\frac{1}{8}}^1 \frac{(x^{\frac{1}{3}} + 2)^2}{x^{\frac{1}{3}}} dx$

Solution:

$$\int_{\frac{1}{8}}^1 \frac{(x^{\frac{1}{3}} + 2)^2}{x^{\frac{1}{3}}} dx$$

$$= 3 \int_{\frac{1}{8}}^1 (x^{\frac{1}{3}} + 2)^2 \cdot \frac{1}{3} x^{-\frac{2}{3}} dx$$

$$= 3 \left[\frac{(x^{\frac{1}{3}} + 2)^3}{3} \right]_{\frac{1}{8}}^1$$

$$= (1 + 2)^3 - \left(\left(\frac{1}{8}\right)^{\frac{1}{3}} + 2 \right)^3$$

$$= 27 - \left(\frac{1}{(2^3)^{\frac{1}{3}}} + 2 \right)^3$$

$$= 27 - \left(\frac{1}{2} + 2 \right)^3 = 27 - \left(\frac{1+4}{2} \right)^3$$

$$= 27 - \frac{125}{8} = \frac{216 - 125}{8} = \frac{91}{8} \quad \text{Ans.}$$

Q.24 $\int_1^3 \frac{x^2 - 2}{x + 1} dx$

Solution:

$$\int_1^3 \frac{x^2 - 2}{x + 1} dx$$

$$= \int_1^3 \left(x - 1 - \frac{1}{x + 1} \right) dx$$

$$= \int_1^3 x dx - \int_1^3 dx - \int_1^3 \frac{dx}{x + 1}$$

$$\begin{aligned} & x + 1 \sqrt{x^2 - 2} \\ & \frac{-x^2 \pm x}{-x - 2} \end{aligned}$$



$$\begin{aligned}
 &= \left[\frac{x^2}{2} \right]_1^3 - [x]_1^3 - [\ln |x+1|]_1^3 \\
 &= \frac{1}{2} (9-1) - (3-1) - [\ln |3+1| - \ln |1+1|] \\
 &= \frac{8}{2} - 2 - (\ln 4 - \ln 2) \\
 &= 4 - 2 - \ln \left(\frac{4}{2} \right) \\
 &= 2 - \ln 2 \quad \text{Ans.}
 \end{aligned}$$

Q.25 $\int_2^3 \frac{3x^2 - 2x + 1}{(x-1)(x^2+1)} dx$

Solution:

$$\begin{aligned}
 &\int_2^3 \frac{3x^2 - 2x + 1}{(x-1)(x^2+1)} dx \\
 &= \int_2^3 \frac{3x^2 - 2x + 1}{x^3 - x^2 + x - 1} dx = [\ln |x^3 - x^2 + x - 1|]_2^3 \\
 &= \ln |27 - 9 + 3 - 1| - \ln |8 - 4 + 2 - 1| \\
 &= \ln 20 - \ln 5 = \ln \frac{20}{5} = \ln 4 \quad \text{Ans.}
 \end{aligned}$$

Q.26 $\int_0^{\frac{\pi}{4}} \frac{\sin x - 1}{\cos^2 x} dx$

Solution:

$$\begin{aligned}
 &\int_0^{\frac{\pi}{4}} \frac{\sin x - 1}{\cos^2 x} dx \\
 &= \int_0^{\frac{\pi}{4}} \left(\frac{\sin x}{\cos^2 x} - \frac{1}{\cos^2 x} \right) dx \\
 &= \int_0^{\frac{\pi}{4}} \sec x \tan x dx - \int_0^{\frac{\pi}{4}} \sec^2 x dx \\
 &= [\sec x]_0^{\pi/4} - [\tan x]_0^{\pi/4}
 \end{aligned}$$



$$\begin{aligned}
 &= \left(\sec \frac{\pi}{4} - \sec 0 \right) - \left(\tan \frac{\pi}{4} - \tan 0 \right) \\
 &= \sqrt{2} - 1 - (1 - 0) = \sqrt{2} - 1 - 1 = \sqrt{2} - 2 \quad \text{Ans.}
 \end{aligned}$$

Q.27 $\int_0^{\frac{\pi}{4}} \frac{1}{1 + \sin x} dx$

Solution:

$$\begin{aligned}
 &\int_0^{\frac{\pi}{4}} \frac{1}{1 + \sin x} dx \\
 &= \int_0^{\frac{\pi}{4}} \left(\frac{1}{1 + \sin x} \times \frac{1 - \sin x}{1 - \sin x} \right) dx \\
 &= \int_0^{\frac{\pi}{4}} \left(\frac{1 - \sin x}{1 - \sin^2 x} \right) dx \\
 &= \int_0^{\frac{\pi}{4}} \frac{1 - \sin x}{\cos^2 x} dx \\
 &= \int_0^{\frac{\pi}{4}} \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x} \right) dx \\
 &= \int_0^{\frac{\pi}{4}} \sec^2 x dx - \int_0^{\frac{\pi}{4}} \sec x \tan x dx \\
 &= [\tan x]_0^{\pi/4} - [\sec x]_0^{\pi/4} \\
 &= \left(\tan \frac{\pi}{4} - \tan 0 \right) - \left(\sec \frac{\pi}{4} - \sec 0 \right) \\
 &= (1 - 0) - (\sqrt{2} - 1) = 1 - \sqrt{2} + 1 = 2 - \sqrt{2} \quad \text{Ans.}
 \end{aligned}$$

Q.28 $\int_0^1 \frac{3x}{\sqrt{4 - 3x}} dx$ (Lhr. Board 2008)

Solution:

$$\int_0^1 \frac{3x}{\sqrt{4 - 3x}} dx$$



$$\begin{aligned}
&= -\int_0^1 \frac{4-3x-4}{\sqrt{4-3x}} dx \\
&= -\int_0^1 \left(\frac{4-3x}{\sqrt{4-3x}} - \frac{4}{\sqrt{4-3x}} \right) dx \\
&= -\int_0^1 \sqrt{4-3x} dx + 4 \int_0^1 \frac{dx}{\sqrt{4-3x}} \\
&= \frac{1}{3} \int_0^1 (4-3x)^{\frac{1}{2}} - 3 dx - \frac{4}{3} \int_0^1 (4-3x)^{-\frac{1}{2}} - 3 dx \\
&= \frac{1}{3} \left[\frac{(4-3x)^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^1 - \frac{4}{3} \left[\frac{(4-3x)^{\frac{1}{2}}}{\frac{1}{2}} \right]_0^1 \\
&= \frac{2}{9} \left[(4-3)^{\frac{3}{2}} - (4-0)^{\frac{3}{2}} \right] - \frac{8}{3} \left[(4-3)^{\frac{1}{2}} - (4-0)^{\frac{1}{2}} \right] \\
&= \frac{2}{9} [1 - (2^2)^{\frac{3}{2}}] - \frac{8}{3} [1 - (2^2)^{\frac{1}{2}}] \\
&= \frac{2}{9} (1-8) - \frac{8}{3} (1-2) \\
&= \frac{2}{9} (-7) - \frac{8}{3} (-1) = \frac{-14}{9} + \frac{8}{3} = \frac{-14+24}{9} = \frac{10}{9} \quad \text{Ans.}
\end{aligned}$$

Q.29

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x (2 + \sin x)} dx$$

Solution:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x (2 + \sin x)} dx$$

Put $\sin x = t$

$$\cos x dx = dt \Rightarrow dx = \frac{dt}{\cos x}$$

$$\text{When } x = \frac{\pi}{6}, \quad t = \frac{1}{2}$$



When $x = \frac{\pi}{2}$, $t = 1$

$$= \int_{\frac{1}{2}}^1 \frac{\cos x}{t(2+t)} \times \frac{dt}{\cos x} = \int_{\frac{1}{2}}^1 \frac{dt}{t(2+t)}$$

Taking

$$\frac{t}{t(2+t)} = \frac{A}{t} + \frac{B}{2+t} \quad \text{—————} \quad (1)$$

Multiplying $t(2+t)$ on both sides in eq. (1)

$$1 = A(2+t) + Bt \quad \text{—————} \quad (2)$$

To find A

Put $t = 0$ in equation (2)

$$1 = A(2+0)$$

$$2A = 1$$

$$A = \frac{1}{2}$$

To find B

Put $2+t = 0$

$t = -2$ in equation (2)

$$1 = B(-2)$$

$$B = \frac{-1}{2}$$

∴ From equation (1)

$$\frac{1}{t(2+t)} = \frac{\frac{1}{2}}{t} + \frac{\frac{-1}{2}}{2+t}$$

Integrate from $\frac{1}{2}$ to 1

$$\int_{\frac{1}{2}}^1 \frac{dt}{t(2+t)} = \frac{1}{2} \int_{\frac{1}{2}}^1 \frac{dt}{t} - \frac{1}{2} \int_{\frac{1}{2}}^1 \frac{dt}{2+t}$$

$$= \frac{1}{2} [\ln |t|]_{\frac{1}{2}}^1 - \frac{1}{2} [\ln |2+t|]_{\frac{1}{2}}^1$$

$$= \frac{1}{2} \left(\ln 1 - \ln \frac{1}{2} \right) - \frac{1}{2} \left(\ln 3 - \ln \left(2 + \frac{1}{2} \right) \right)$$



$$\begin{aligned}
 &= \frac{1}{2} \left[-\ln \frac{1}{2} - \ln 3 + \ln \left(\frac{4+1}{2} \right) \right] \\
 &= \frac{1}{2} \left[-\ln \frac{1}{2} - \ln 3 + \ln \frac{5}{2} \right] \\
 &= \frac{1}{2} \ln \left(\frac{5/2}{\frac{1}{2} \times 3} \right) \\
 &= \frac{1}{2} \ln \left(\frac{5}{3} \right) \quad \text{Ans}
 \end{aligned}$$

Q.30 $\int_0^{\frac{\pi}{2}} \frac{\sin x}{(1 + \cos x)(2 + \cos x)} dx$

Solution:

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{(1 + \cos x)(2 + \cos x)} dx$$

Put

$$\cos x = t$$

$$-\sin x \, dx = dt$$

$$dx = \frac{dt}{-\sin x}$$

When $x = 0$, $t = 1$

When $t = \frac{\pi}{2}$, $t = 0$

$$= \int_1^0 \frac{\sin x}{(1+t)(2+t)} \times \frac{dt}{-\sin x} = \int_0^1 \frac{dt}{(1+t)(2+t)}$$

Taking

$$\frac{1}{(1+t)(2+t)} = \frac{A}{1+t} + \frac{B}{2+t} \quad \text{--- (1)}$$

Multiplying $(1+t)(2+t)$ on both sides in eq. (1)

$$1 = A(2+t) + B(1+t) \quad \text{--- (2)}$$

To find A

Put

$$1+t = 0$$

$$t = -1 \text{ in equation (2)}$$

$$1 = A(2-1)$$



$$\boxed{A = 1}$$

To find B

Put $2 + t = 0$

$t = -2$ in equation (2)

$$1 = B(1 - 2)$$

$$-B = 1$$

$$\boxed{B = -1}$$

∴ From equation (1)

$$\frac{1}{(1+t)(2+t)} = \frac{1}{1+t} + \frac{-1}{2+t}$$

Integrate from 0 to 1

$$\begin{aligned} \int_0^1 \frac{dt}{(1+t)(2+t)} &= \int_0^1 \frac{dt}{1+t} - \int_0^1 \frac{dt}{2+t} \\ &= [\ln |1+t|]_0^1 - [\ln |2+t|]_0^1 \\ &= (\ln 2 - \ln 1) - (\ln 3 - \ln 2) \\ &= \ln 2 - \ln 3 + \ln 2 \\ &= \ln \frac{2 \times 2}{3} = \ln \frac{4}{3} \quad \text{Ans} \end{aligned}$$



EXERCISE 3.7

Q.1 Find the area between the x-axis and the curve $y = x^2 + 1$ from $x = 1$ to $x = 2$
(Lhr. Board 2005, 2008)

Solution:

$$y = x^2 + 1 \quad \text{from } x = 1 \text{ to } x = 2$$

$$\begin{aligned} \text{Required area} &= \int_1^2 y \, dx \\ &= \int_1^2 (x^2 + 1) \, dx \\ &= \int_1^2 x^2 \, dx + \int_1^2 1 \, dx \\ &= \left[\frac{x^3}{3} \right]_1^2 + [x]_1^2 \end{aligned}$$



$$= \frac{1}{3} (8 - 1) + (2 - 1) = \frac{7}{3} + 1 = \frac{7 + 3}{3} = \frac{10}{3} \text{ sq. units} \quad \text{Ans.}$$

Q.2 Find the area, above the x-axis and under the curve $y = 5 - x^2$ from $x = -1$ to $x = 2$.
(Lhr. Board 2011)

Solution:

$$y = 5 - x^2 \quad \text{from} \quad x = -1 \quad \text{to} \quad x = 2$$

$$\begin{aligned} \text{Required area} &= \int_a^b y \, dx \\ &= \int_{-1}^2 (5 - x^2) \, dx \\ &= 5 \int_{-1}^2 dx - \int_{-1}^2 x^2 \, dx \\ &= 5[x]_{-1}^2 - \left[\frac{x^3}{3} \right]_{-1}^2 \\ &= 5(2 + 1) - \frac{1}{3}(8 + 1) = 5(3) - \frac{1}{3}(9) = 15 - 3 = 12 \text{ sq. units} \end{aligned}$$

Q.3 Find the area below the curve $y = 3\sqrt{x}$ and above the x-axis between $x = 1$ and $x = 4$.

Solution:

$$y = 3\sqrt{x}$$

$$\begin{aligned} \text{Required area} &= \int_a^b y \, dx \\ &= \int_1^4 3\sqrt{x} \, dx \\ &= 3 \int_1^4 x^{1/2} \, dx \\ &= 3 \left[\frac{x^{3/2}}{3/2} \right]_1^4 \\ &= 2 \left[\frac{3}{2} x^{1/2} \right]_1^4 \\ &= 2 \left[(2^2)^{1/2} - 1 \right] = 2(8 - 1) = 2(7) = 14 \text{ Sq. units} \end{aligned}$$



Q.4 Find the area bounded by cos function from $x = \frac{-\pi}{2}$ to $x = \frac{\pi}{2}$. (Guj. Board 2008)

Solution:

$$y = \cos x$$

$$\text{Required area} = \int_a^b y \, dx = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos x \, dx$$

$$= [\sin x]_{\frac{-\pi}{2}}^{\frac{\pi}{2}}$$

$$= \sin \frac{\pi}{2} - \sin \left(\frac{-\pi}{2} \right)$$

$$= 1 + 1$$

$$= 2 \text{ Sq. units} \quad \text{Ans.}$$

Q.5 Find the area between the x-axis and the curve $y = 4x - x^2$ (Lhr. Board 2009, Guj Board 2005, 2008)

Solution:

$$y = 4x - x^2$$

To find the limits

$$\text{Put } y = 0$$

$$4x - x^2 = 0$$

$$x(4 - x) = 0$$

Either

$$x = 0 \quad \text{or} \quad 4 - x = 0$$

$$x = 4$$

The curve cuts the x-axis at (0, 0) and (4, 0)

$$y \geq 0 \quad \text{for} \quad 0 \leq x \leq 4$$

That is, the area in the interval [0, 4] is above the x-axis.

$$\begin{aligned} \text{Required Area} &= \int_a^b y \, dx \\ &= \int_0^4 (4x - x^2) \, dx \\ &= 4 \int_0^4 x \, dx - \int_0^4 x^2 \, dx \end{aligned}$$



$$\begin{aligned}
 &= 4 \left[\frac{x^2}{2} \right]_0^4 - \left[\frac{x^3}{3} \right]_0^4 \\
 &= 2(16 - 0) - \frac{1}{3}(64 - 0) \\
 &= 32 - \frac{64}{3} = \frac{96 - 64}{3} \\
 &= \frac{32}{3} \text{ Sq. units} \quad \text{Ans.}
 \end{aligned}$$

Q.6 Determine the area bounded by the parabola $y = x^2 + 2x - 3$ and the x-axis

Solution:

$$y = x^2 + 2x - 3$$

To find the limits

Put

$$y = 0$$

$$x^2 + 2x - 3 = 0$$

$$x^2 + 3x - x - 3 = 0$$

$$x(x + 3) - 1(x + 3) = 0$$

$$(x + 3)(x - 1) = 0$$

Either

$$x + 3 = 0 \quad \text{or} \quad x - 1 = 0$$

$$x = -3 \quad \quad \quad x = 1$$

The curve cuts the x-axis at $(-3, 0)$ and $(1, 0)$

$$y \leq 0 \quad \text{for} \quad -3 \leq x \leq 1$$

That is, the area in the interval $[-3, 1]$ is below the x-axis

$$\begin{aligned}
 \text{Required Area} &= - \int_a^b y \, dx \\
 &= - \int_{-3}^1 (x^2 + 2x - 3) \, dx \\
 &= - \int_{-3}^1 x^2 \, dx - 2 \int_{-3}^1 x \, dx + 3 \int_{-3}^1 dx \\
 &= - \left[\frac{x^3}{3} \right]_{-3}^1 - 2 \left[\frac{x^2}{2} \right]_{-3}^1 + 3 [x]_{-3}^1 \\
 &= \frac{-1}{3} (1 + 27) - (1 - 9) + 3 (1 + 3)
 \end{aligned}$$



$$\begin{aligned}
 &= \frac{-28}{3} - (-8) + 3(4) \\
 &= \frac{-28}{3} + 8 + 12 = \frac{-28}{3} + 20 \\
 &= \frac{-28 + 60}{3} = \frac{32}{3} \text{ Sq. units} \quad \text{Ans.}
 \end{aligned}$$

Q.7 Find the area bounded by the curve $y = x^3 + 1$, the x-axis and line $x = 2$.

Solution:

$$y = x^3 + 1$$

To find the limits

Put

$$\begin{aligned}
 y &= 0 \\
 x^3 + 1 &= 0 \\
 (x)^3 + (1)^3 &= 0 \\
 (x + 1)(x^2 - x + 1) &= 0
 \end{aligned}$$

Either

$$\begin{aligned}
 x + 1 &= 0 & \text{or} & & x^2 - x + 1 &= 0 \\
 x &= -1 & & & \text{Neglecting because it has imaginary roots,}
 \end{aligned}$$

$$\begin{aligned}
 \text{Required Area} &= \int_a^b y \, dx \\
 &= \int_{-1}^2 (x^3 + 1) \, dx \\
 &= \int_{-1}^2 x^3 \, dx + \int_{-1}^2 dx \\
 &= \left[\frac{x^4}{4} \right]_{-1}^2 + [x]_{-1}^2 \\
 &= \frac{1}{4} (16 - 1) + (2 + 1) \\
 &= \frac{15}{4} + 3 \\
 &= \frac{15 + 12}{4} = \frac{27}{4} \text{ Sq. units} \quad \text{Ans.}
 \end{aligned}$$

Q.8 Find the area bounded by the curve $y = x^3 - 4x$ and the x-axis.

Solution:

$$y = x^3 - 4x$$



To find the limits

Put

$$\begin{aligned}y &= 0 \\x^3 - 4x &= 0 \\x(x^2 - 4) &= 0 \\x(x+2)(x-2) &= 0\end{aligned}$$

Either

$$\begin{aligned}x = 0 \quad \text{or} \quad x + 2 = 0 \quad \text{or} \quad x - 2 = 0 \\x = -2 \quad \quad \quad x = 2\end{aligned}$$

The curve cuts the x-axis at $(-2, 0)$, $(0, 0)$ and $(2, 0)$

$$y \geq 0 \quad \text{for} \quad -2 \leq x \leq 0$$

That is, the area in the interval $[-2, 0]$ is above the x-axis.

$$y \leq 0 \quad \text{for} \quad 0 \leq x \leq 2$$

That is, the area in the interval $[0, 2]$ lies below the x-axis

$$\begin{aligned}\text{Required Area} &= \int_{-2}^0 y dx - \int_0^2 y dx \\&= \int_{-2}^0 (x^3 - 4x) dx - \int_0^2 (x^3 - 4x) dx \\&= \int_{-2}^0 x^3 dx - 4 \int_{-2}^0 x dx - \int_0^2 x^3 dx + 4 \int_0^2 x dx \\&= \left[\frac{x^4}{4} \right]_{-2}^0 - 4 \left[\frac{x^2}{2} \right]_{-2}^0 - \left[\frac{x^4}{4} \right]_0^2 + 4 \left[\frac{x^2}{2} \right]_0^2 \\&= \frac{1}{4} (0 - 16) - 2 (0 - 4) - \frac{1}{4} (16 - 0) + 2 (4 - 0) \\&= \frac{-16}{4} - 2 (-4) - \frac{1}{4} (16) + 8 \\&= -4 + 8 - 4 + 8 \\&= 8 \text{ Sq. units Ans.}\end{aligned}$$

Q.9 Find the area between the curve $y = x(x-1)(x+1)$ and the x-axis.

Solution:

$$y = x(x-1)(x+1)$$

To find the limits

Put

$$\begin{aligned}y &= 0 \\x(x-1)(x+1) &= 0\end{aligned}$$



Either

$$\begin{array}{llll} x = 0 & \text{or} & x - 1 = 0 & \text{or} & x + 1 = 0 \\ & & x = 1 & & x = -1 \end{array}$$

The curve cuts the x-axis at $(-1, 0)$, $(0, 0)$ and $(1, 0)$

$$y \geq 0 \quad \text{for} \quad -1 \leq x \leq 0$$

That is, the area in the interval $[-1, 0]$ lies above the x-axis.

$$y \leq 0 \quad \text{for} \quad 0 \leq x \leq 1$$

That is, the area in the interval $[0, 1]$ lies below the x-axis.

$$\begin{aligned} \text{Required Area} &= \int_{-1}^0 y dx - \int_0^1 y dx \\ &= \int_{-1}^0 x(x-1)(x+1) dx - \int_0^1 x(x-1)(x+1) dx \\ &= \int_{-1}^0 x(x^2-1) dx - \int_0^1 x(x^2-1) dx \\ &= \int_{-1}^0 (x^3-x) dx - \int_0^1 (x^3-x) dx \\ &= \int_{-1}^0 x^3 dx - \int_{-1}^0 x dx - \int_0^1 x^3 dx + \int_0^1 x dx \\ &= \left[\frac{x^4}{4} \right]_{-1}^0 - \left[\frac{x^2}{2} \right]_{-1}^0 - \left[\frac{x^4}{4} \right]_0^1 + \left[\frac{x^2}{2} \right]_0^1 \\ &= \frac{1}{4}(0-1) - \frac{1}{2}(0-1) - \frac{1}{4}(1-0) + \frac{1}{2}(1-0) \\ &= \frac{-1}{4} + \frac{1}{2} - \frac{1}{4} + \frac{1}{2} \\ &= \frac{-1+2-1+2}{4} = \frac{2}{4} = \frac{1}{2} \text{ Sq. units} \quad \text{Ans.} \end{aligned}$$

Q.10 Find the area above the x-axis bounded by the curve $y^2 = 3 - x$ from $x = -1$ to $x = 2$.

Solution:

$$y^2 = 3 - x \quad \text{from} \quad x = -1 \quad \text{to} \quad x = 2$$

$$y = \sqrt{3 - x}$$

$$\text{Required Area} = \int_a^b y dx$$



$$\begin{aligned}
 &= \int_{-1}^2 \sqrt{3-x} \, dx = -\int_{-1}^2 (3-x)^{\frac{1}{2}} \cdot -dx \\
 &= -\left[\frac{(3-x)^{\frac{3}{2}}}{\frac{3}{2}} \right]_{-1}^2 \\
 &= \frac{-2}{3} \left[(3-2)^{\frac{3}{2}} - (3+1)^{\frac{3}{2}} \right] \\
 &= \frac{-2}{3} \left[(1)^{\frac{3}{2}} - (4)^{\frac{3}{2}} \right] \\
 &= \frac{-2}{3} \left[1 - (2^2)^{\frac{3}{2}} \right] \\
 &= \frac{-2}{3} (1-8) \\
 &= \frac{-2}{3} (-7) = \frac{14}{3} \text{ Sq. units} \quad \text{Ans.}
 \end{aligned}$$

Q.11 Find the area between the x-axis and the curve $y = \cos \frac{1}{2}x$ from $x = -\pi$ to π .

Solution:

$$y = \cos \frac{1}{2}x \quad \text{from } x = -\pi \text{ to } x = \pi$$

$$\begin{aligned}
 \text{Required Area} &= \int_a^b y \, dx \\
 &= \int_{-\pi}^{\pi} \cos \frac{1}{2}x \, dx \\
 &= \left[\frac{\sin \frac{x}{2}}{\frac{1}{2}} \right]_{-\pi}^{\pi} \\
 &= 2 \left[\sin \frac{\pi}{2} - \sin \left(-\frac{\pi}{2} \right) \right] \\
 &= 2(1+1) \\
 &= 2(2) \\
 &= 4 \text{ Sq. units} \quad \text{Ans.}
 \end{aligned}$$



Q.12 Find the area between the x-axis and the curve $y = \sin 2x$ from $x = 0$ to $x = \frac{\pi}{3}$.

Solution:

$$y = \sin 2x \quad \text{from } x = 0 \quad \text{to } x = \frac{\pi}{3}$$

$$\begin{aligned} \text{Required Area} &= \int_a^b y dx \\ &= \int_0^{\frac{\pi}{3}} \sin 2x \, dx \end{aligned}$$

$$= \left[\frac{-\cos 2x}{2} \right]_0^{\frac{\pi}{3}}$$

$$= \frac{-1}{2} \left[\cos 2 \left(\frac{\pi}{3} \right) - \cos 2(0) \right]$$

$$= \frac{-1}{2} \left[\frac{-1}{2} - 1 \right]$$

$$= \frac{-1}{2} \left(\frac{-1-2}{2} \right)$$

$$= \frac{-1}{2} \left(\frac{-3}{2} \right)$$

$$= \frac{3}{4} \text{ Sq. units} \quad \text{Ans.}$$

Q.13 Find the area between the x-axis and the curve $y = \sqrt{2ax - x^2}$ when $a > 0$.

Solution:

$$y = \sqrt{2ax - x^2}$$

To find the limits

Put

$$y = 0$$

$$\sqrt{2ax - x^2} = 0$$

$$2ax - x^2 = 0$$

$$x(2a - x) = 0$$

Either

$$x = 0$$

or

$$2a - x = 0$$

$$x = 2a$$



$$\begin{aligned}
 \text{Required Area} &= \int_0^b y dx \\
 &= \int_0^{2a} \sqrt{2ax - x^2} dx \\
 &= \int_0^{2a} \sqrt{a^2 - a^2 + 2ax - x^2} dx \\
 &= \int_0^{2a} \sqrt{a^2 - (a^2 - 2ax + x^2)} dx \\
 &= \int_0^{2a} \sqrt{a^2 - (a - x)^2} dx
 \end{aligned}$$

Put $a - x = a \sin \theta$
 $-dx = a \cos \theta d\theta$
 $dx = -a \cos \theta d\theta$

When $x = 0$, $a - 0 = a \sin \theta$
 $\sin \theta = \frac{a}{a} = 1$

$$\theta = \frac{\pi}{2}$$

When $x = 2a$, $a - 2a = a \sin \theta$
 $-a = a \sin \theta$
 $\sin \theta = \frac{-a}{a} = -1$
 $\theta = \frac{-\pi}{2}$

$$\begin{aligned}
 &= \int_{\frac{\pi}{2}}^{\frac{-\pi}{2}} \sqrt{a^2 - a^2 \sin^2 \theta} (-a \cos \theta) d\theta \\
 &= -a \int_{\frac{\pi}{2}}^{\frac{-\pi}{2}} \sqrt{a^2 (1 - \sin^2 \theta)} \cos \theta d\theta \\
 &= a \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} a \sqrt{\cos^2 \theta} \cdot \cos \theta d\theta
 \end{aligned}$$



$$\begin{aligned}
&= a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta \cdot \cos\theta \, d\theta \\
&= a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta \, d\theta \\
&= a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1 + \cos 2\theta}{2} \right) d\theta \\
&= \frac{a^2}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \cos 2\theta) \, d\theta \\
&= \frac{a^2}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta + \frac{a^2}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 2\theta \, d\theta \\
&= \frac{a^2}{2} \left[\theta \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{a^2}{2} \left[\frac{\sin 2\theta}{2} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \frac{a^2}{2} \left[\frac{\pi}{2} + \frac{\pi}{2} \right] + \frac{a^2}{4} \left[\sin 2 \left(\frac{\pi}{2} \right) - \sin 2 \left(\frac{-\pi}{2} \right) \right] \\
&= \frac{a^2}{2} \left(\frac{\pi + \pi}{2} \right) + \frac{a^2}{4} (0 + 0) \\
&= \frac{a^2}{2} \left(\frac{2\pi}{2} \right) \\
&= \frac{a^2\pi}{2} \text{ Sq. units} \quad \text{Ans.}
\end{aligned}$$



EXERCISE 3.8

Q.1 Check that each of the following equations written against the differential equation in its solution.

- | | |
|--|-----------------------------|
| (i) $x \frac{dy}{dx} = 1 + y$ | $y = cx - 1$ |
| (ii) $x^2 (2y + 1) \frac{dy}{dx} - 1 = 0$ | $y^2 + y = c - \frac{1}{x}$ |
| (iii) $y \frac{dy}{dx} - e^{2x} = 1$ | $y^2 = e^{2x} + 2x + c$ |
| (iv) $\frac{1}{x} \frac{dy}{dx} - 2y = 0$ | $y = ce^{x^2}$ |
| (v) $\frac{dy}{dx} = \frac{y^2 + 1}{e^{-x}}$ | $y = \tan(e^x + c)$ |

Solution:

(i) $x \frac{dy}{dx} = 1 + y$ ————— (1) (Lhr. Board 2007)

$$y = cx - 1$$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = c$$

Put in equation (1)

$$xc = 1 + e^x - 1$$

$$xc = e^x$$

Which is true

$\therefore y = cx - 1$ is the solution of $x \frac{dy}{dx} = 1 + y$

(ii) $x^2 (2y + 1) \frac{dy}{dx} - 1 = 0$ ————— (1)

$$y^2 + y = c - \frac{1}{x}$$

Diff. w.r.t. 'x'

$$2y \frac{dy}{dx} + \frac{dy}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} (2y + 1) = \frac{1}{x^2}$$

$$\frac{dy}{dx} = \frac{1}{x^2 (2y + 1)}$$

Put in equation (1)

$$x^2 (2y + 1) \frac{1}{x^2 (2y + 1)} - 1 = 0$$

$$1 - 1 = 0$$

$$0 = 0$$



Which is true

$\therefore y^2 + y = c - \frac{1}{x}$ is the solution of $x^2 (2y + 1) \frac{dy}{dx} - 1 = 0$

(iii) $y \frac{dy}{dx} - e^{2x} = 1$ ——— (1)

$$y^2 = e^{2x} + 2x + c$$

Diff. W.r.t. 'x'

$$2y \frac{dy}{dx} = 2e^{2x} + 2$$

$$\frac{dy}{dx} = \frac{2e^{2x} + 2}{2y} = \frac{2(e^{2x} + 1)}{2y} = \frac{e^{2x} + 1}{y}$$

Put in equation (1)

$$y \left(\frac{e^{2x} + 1}{y} \right) - e^{2x} = 1$$

$$e^{2x} + 1 - e^{2x} = 1$$

$$1 = 1$$

Which is true

$\therefore y^2 = e^{2x} + 2x + c$ is the solution of $y \frac{dy}{dx} - e^{2x} = 1$

(iv) $\frac{1}{x} \frac{dy}{dx} - 2y = 0$ ——— (1)

$$y = ce^{x^2}$$

diff. w.r.t. 'x'

$$\frac{dy}{dx} = cxe^{x^2} \cdot 2x$$

$$= 2cxe^{x^2}$$

Put in equation (1)

$$\frac{1}{x} \cdot 2cxe^{x^2} - 2(ce^{x^2}) = 0$$

$$2ce^{x^2} - 2ce^{x^2} = 0$$

$$0 = 0$$

Which is true

$\therefore y = ce^{x^2}$ is the solution of $\frac{1}{x} \frac{dy}{dx} - 2y = 0$

(v) $\frac{dy}{dx} = \frac{y^2 + 1}{e^{-x}}$ ——— (1)

$$y = \tan(e^x + c)$$



Diff. w.r.t. 'x'

$$\begin{aligned}\frac{dy}{dx} &= \sec^2(e^x + c) \cdot e^x \\ &= \frac{1 + \tan^2(e^x + c)}{e^{-x}} \\ &= \frac{1 + y^2}{e^{-x}}\end{aligned}$$

∴ From equation (1)

$$\frac{y^2 + 1}{e^{-x}} = \frac{y^2 + 1}{e^{-x}}$$

Which is true

∴ $y = \tan(e^x + c)$ is the solution of $\frac{dy}{dx} = \frac{y^2 + 1}{e^{-x}}$

Q.2 $\frac{dy}{dx} = -y$

Solution:

$$\frac{dy}{dx} = -y$$

Separate variables

$$\frac{dy}{y} = -dx$$

Integrate

$$\int \frac{dy}{y} = - \int dx$$

$$\ln y = -x + \ln c_1$$

$$e^{\ln y} = e^{-x + \ln c_1}$$

$$y = e^{-x} \cdot e^{\ln c_1}$$

$$\therefore y = e^{\ln y}$$

$$y = ce^{-x}$$

$$\therefore e^{\ln c_1} = c \quad \text{Ans.}$$

Q.3 $y dx + x dy = 0$ (Guj. Board 2006)

Solution:

$$y dx + x dy = 0$$

Separate variables

$$y dx = -x dy$$

$$\frac{dy}{y} = -\frac{dx}{x}$$



Integrate

$$\int \frac{dy}{y} = - \int \frac{dx}{x}$$

$$\ln y = - \ln x + \ln c$$

$$\ln y = \ln \frac{c}{x}$$

$$y = \frac{c}{x}$$

$$\boxed{xy = c}$$

Ans

Q.4 $\frac{dy}{dx} = \frac{1-x}{y}$ (Lhr. Board 2008)

Solution:

$$\frac{dy}{dx} = \frac{1-x}{y}$$

Separate variables

$$y dy = (1-x) dx$$

Integrate

$$\int y dy = \int dx - \int x dx$$

$$\frac{y^2}{2} = x - \frac{x^2}{2} + c$$

$$y^2 = 2\left(x - \frac{x^2}{2}\right) + 2c$$

$$y^2 = 2x - x^2 + c$$

$$y^2 = x(2-x) + c$$

$$\therefore 2c_1 =$$

Ans.

Q.5 $\frac{dy}{dx} = \frac{y}{x^2}, (y > 0)$ (Guj. Board 2008)

Solution:

$$\frac{dy}{dx} = \frac{y}{x^2}, (y > 0)$$

Separate variables

$$\frac{dy}{y} = \frac{dx}{x^2}$$

Integrate

$$\int \frac{dy}{y} = \int \frac{dx}{x^2}$$



$$\ln y = \frac{x^{-1}}{-1} + \ln c$$

$$\ln y = \frac{-1}{x} + \ln c$$

$$e^{\ln y} = e^{\frac{-1}{x} + \ln c}$$

$$y = e^{\frac{-1}{x}} e^{\ln c}$$

$$y = ce^{\frac{-1}{x}}$$

$$\therefore y = e^{\ln y}$$

Ans.

Q.6 $\sin y \operatorname{cosec} x \frac{dy}{dx} = 1$ (Guj. Board 2005, 2008)

Solution:

$$\sin y \operatorname{cosec} x \frac{dy}{dx} = 1$$

Separate variables

$$\sin y dy = \frac{dx}{\operatorname{cosec} x}$$

$$\sin y dy = \sin x dx$$

Integrate

$$\int \sin y dy = \int \sin x dx$$

$$-\cos y = -\cos x + c$$

$$\cos y = \cos x + c$$

Ans.

Q.7 $x dy + y(x-1) dx = 0$ (Lhr. Board 2007)

Solution:

$$x dy + y(x-1) dx = 0$$

Separate variables

$$x dy = -y(x-1) dx$$

$$\frac{dy}{y} = \frac{-x+1}{x} dx$$

$$\frac{dy}{y} = \left(\frac{-x}{x} + \frac{1}{x} \right) dx$$

Integrate

$$\int \frac{dy}{y} = \int dx + \int \frac{dx}{x}$$

$$\ln y = -x + \ln x + \ln c$$

$$e^{\ln y} = e^{-x + \ln x + \ln c}$$

$$\therefore y = e^{\ln y}$$



$$y = e^{-x} \cdot e^{\ln x} \cdot e^{\ln c}$$

$$\boxed{y = cxe^{-x}} \quad \text{Ans.}$$

Q.8 $\frac{x^2+1}{y+1} = \frac{x}{y} \cdot \frac{dy}{dx}, (x, y > 0)$ (Guj. Board 2006)

Solution:

$$\frac{x^2+1}{y+1} = \frac{x}{y} \cdot \frac{dy}{dx}$$

Separate variables

$$\frac{x^2+1}{x} dx = \frac{y+1}{y} dy$$

$$\left(\frac{y}{y} + \frac{1}{y}\right) dy = \left(\frac{x^2}{x} + \frac{1}{x}\right) dx$$

$$dy + \frac{dy}{y} = x dx + \frac{dx}{x}$$

Integrate

$$\int dy + \int \frac{dy}{y} = \int x dx + \int \frac{dx}{x}$$

$$y + \ln y = \frac{x^2}{2} + \ln x + \ln c$$

$$e^{y+\ln y} = e^{\frac{x^2}{2} + \ln x + \ln c}$$

$$e^y \cdot e^{\ln y} = e^{\frac{x^2}{2}} \cdot e^{\ln x} \cdot e^{\ln c}$$

$$ye^y = cxe^{\frac{x^2}{2}} \quad \text{Ans.}$$

$$\therefore y = e^{\ln y}$$

Q.9 $\frac{1}{x} \frac{dy}{dx} = \frac{1}{2} (1+y^2)$ (Lhr. Board 2008)

Solution:

$$\frac{1}{x} \frac{dy}{dx} = \frac{1}{2} (1+y^2)$$

Separate variables

$$\frac{dy}{1+y^2} = \frac{x}{2} dx$$

Integrate

$$\int \frac{dy}{1+y^2} = \frac{1}{2} \int x dx$$

$$\tan^{-1} y = \frac{1}{2} \cdot \frac{x^2}{2} + c$$



$$\tan^{-1}y = \frac{x^2}{4} + c \quad \text{Ans.}$$

Q.10 $2x^2y \frac{dy}{dx} = x^2 - 1$

Solution:

$$2x^2y \frac{dy}{dx} = x^2 - 1$$

Separate variables

$$2ydy = \frac{x^2 - 1}{x^2} dx$$

$$2ydy = \left(\frac{x^2}{x^2} - \frac{1}{x^2} \right) dx$$

$$2ydy = dx - x^{-2} dx$$

Integrate

$$2 \int ydy = \int dx - \int x^{-2} dx$$

$$\frac{2y^2}{2} = x - \frac{x^{-1}}{-1} + c$$

$$y^2 = x + \frac{1}{x} + c \quad \text{Ans.}$$

Q.11 $\frac{dy}{dx} + \frac{2xy}{2y+1} = x$ **(Lhr. Board 2009 (S))**

Solution:

$$\frac{dy}{dx} + \frac{2xy}{2y+1} = x$$

Separate variables

$$\frac{dy}{dx} = x - \frac{2xy}{2y+1}$$

$$\frac{dy}{dx} = x \left(1 - \frac{2y}{2y+1} \right)$$

$$\frac{dy}{dx} = x \left(\frac{2y+1-2y}{2y+1} \right)$$

$$\frac{dy}{dx} = \frac{x}{2y+1}$$

$$(2y+1) dy = x dx$$

Integrate

$$2 \int ydy + \int dy = \int x dx$$



$$\begin{aligned}\frac{2y^2}{2} + y &= \frac{x^2}{2} + c \\ y^2 + y &= \frac{x^2}{2} + c \\ y(y+1) &= \frac{x^2}{2} + c \quad \text{Ans.}\end{aligned}$$

Q.12 $(x^2 - yx^2) \frac{dy}{dx} + y^2 + xy^2 = 0$ (Lhr. Board 2006, 2011)

Solution:

$$(x^2 - yx^2) \frac{dy}{dx} + y^2 + xy^2 = 0$$

Separate variables

$$\begin{aligned}x^2(1-y) \frac{dy}{dx} &= -y^2 - xy^2 \\ x^2(1-y) \frac{dy}{dx} &= -y^2(1+x) \\ \left(\frac{1-y}{-y^2}\right) dy &= \left(\frac{1+x}{x^2}\right) dx \\ \left(\frac{-1}{y^2} + \frac{y}{y^2}\right) dy &= \left(\frac{1}{x^2} + \frac{x}{x^2}\right) dx \\ -y^{-2} dy + \frac{dy}{y} &= x^{-2} dx + \frac{dx}{x}\end{aligned}$$

Integrate

$$\begin{aligned}-\int y^{-2} dy + \int \frac{dy}{y} &= \int x^{-2} dx + \int \frac{dx}{x} \\ -\frac{y^{-1}}{-1} + \ln y &= \frac{x^{-1}}{-1} + \ln x + c \\ \frac{1}{y} + \ln y &= \frac{-1}{x} + \ln x + c \\ \ln y + \frac{1}{y} &= \ln x - \frac{1}{x} + c \quad \text{Ans}\end{aligned}$$

Q.13 $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$ (Lhr. Board 2005)

Solution:

$$\begin{aligned}\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy &= 0 \\ \sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy &= 0\end{aligned}$$

Separate variables

$$\sec^2 y \tan x \, dy = -\sec^2 x \tan y \, dx$$



$$\frac{\sec^2 y}{\tan y} dy = \frac{-\sec^2 x}{\tan x} dx$$

Integrate

$$\int \frac{\sec^2 y}{\tan y} dy = - \int \frac{\sec^2 x}{\tan x} dx$$

$$\ln (\tan y) = - \ln (\tan x) + \ln c$$

$$\ln (\tan y) = \ln \frac{c}{\tan x}$$

$$\tan y = \frac{c}{\tan x}$$

$$\tan x \tan y = c \quad \text{Ans.}$$

Q.14 $\left(y - x \frac{dy}{dx}\right) = 2\left(y^2 + \frac{dy}{dx}\right)$

Solution:

$$\left(y - x \frac{dy}{dx}\right) = 2\left(y^2 + \frac{dy}{dx}\right)$$

Separate variables

$$y - x \frac{dy}{dx} = 2y^2 + 2 \frac{dy}{dx}$$

$$y - 2y^2 = 2 \frac{dy}{dx} + x \frac{dy}{dx}$$

$$\frac{dy}{dx} (2 + x) = y (1 - 2y)$$

$$\frac{dy}{y(1 - 2y)} = \frac{dx}{2 + x}$$

Integrate

$$\int \frac{dy}{y(1 - 2y)} = \int \frac{dx}{2 + x}$$

$$I = \ln (2 + x) + c_1 \quad \text{—————} \quad (1)$$

$$I = \int \frac{dy}{y(1 - 2y)}$$

Let

$$\frac{1}{y(1 - 2y)} = \frac{A}{y} + \frac{B}{1 - 2y} \quad \text{—————} \quad (2)$$

Multiplying $y(1 - 2y)$ on both sides in eq. (2)

$$1 = A(1 - 2y) + By \quad \text{—————} \quad (3)$$



To find A

Put $y = 0$ in eq. (3)

$$\boxed{1 = A}$$

To find B

Put

$$1 - 2y = 0$$

$$2y = 1$$

$$y = \frac{1}{2} \text{ in eq. (3)}$$

$$1 = B \left(\frac{1}{2} \right)$$

$$\boxed{B = 2}$$

∴ From equation (2)

$$\frac{1}{y(1-2y)} = \frac{1}{y} + \frac{2}{1-2y}$$

Integrate

$$\int \frac{dy}{y(1-2y)} = \int \frac{dy}{y} + \int \frac{2}{1-2y} dy$$

$$I = \ln y - \int \frac{2}{2y-1} dy$$

$$= \ln y - \ln(2y-1) + c_2$$

Put in eq. (1)

$$\ln y - \ln(2y-1) + c_2 = \ln(2+x) + c_1$$

$$\ln \frac{y}{2y-1} = \ln(2+x) + c_1 - c_2$$

$$\ln \frac{y}{2y-1} = \ln(2+x) + \ln c, \text{ where } c_1 - c_2 = \ln c$$

$$\ln \frac{y}{2y-1} = \ln c(2+x)$$

$$\frac{y}{2y-1} = c(x+2) \quad \text{Ans.}$$

Q.15 $1 + \cos x \tan y \frac{dy}{dx} = 0$

Solution:

$$1 + \cos x \tan y \frac{dy}{dx} = 0$$



Separate variables

$$\cos x \tan y \frac{dy}{dx} = -1$$

$$\tan y \, dy = \frac{-1}{\cos x} \, dx$$

$$\frac{-\sin y}{\cos y} \, dy = \sec x \, dx$$

Integrate

$$\int \frac{-\sin y}{\cos y} \, dy = \int \sec x \, dx$$

$$\ln (\cos y) = \ln (\sec x + \tan x) + \ln c$$

$$\ln (\cos y) = \ln c (\sec x + \tan x)$$

$$\cos y = c(\sec x + \tan x) \quad \text{Ans.}$$

Q.16 $y - x \frac{dy}{dx} = 3 \left(1 + x \frac{dy}{dx} \right)$

Solution:

$$y - x \frac{dy}{dx} = 3 \left(1 + x \frac{dy}{dx} \right)$$

Separate variables

$$y - x \frac{dy}{dx} = 3 + 3x \frac{dy}{dx}$$

$$y - 3 = 3x \frac{dy}{dx} + x \frac{dy}{dx}$$

$$y - 3 = (3x + x) \frac{dy}{dx}$$

$$\frac{dx}{4x} = \frac{dy}{y - 3}$$

Integrate

$$\int \frac{dy}{y - 3} = \frac{1}{4} \int \frac{dx}{x}$$

$$\ln (y - 3) = \frac{1}{4} \ln x + \ln c$$

$$\ln (y - 3) = \frac{1}{4} \ln x^{\frac{1}{4}} + \ln c$$

$$\ln (y - 3) = \ln cx^{\frac{1}{4}}$$

$$y - 3 = cx^{\frac{1}{4}} \quad y = 3 + cx^{\frac{1}{4}} \quad \text{Ans.}$$



Q.17 $\sec x + \tan y \frac{dy}{dx} = 0$

Solution:

$$\sec x + \tan y \frac{dy}{dx} = 0$$

Separate variables

$$\tan y \frac{dy}{dx} = -\sec x$$

$$-\tan y \, dy = \sec x \, dx$$

$$\frac{-\sin y}{\cos y} \, dy = \sec x \, dx$$

Integrate

$$\int \frac{-\sin y}{\cos y} \, dy = \int \sec x \, dx$$

$$\ln(\cos y) = \ln(\sec x + \tan x) + \ln c$$

$$\ln(\cos y) = \ln c (\sec x + \tan x)$$

$$\cos y = c(\sec x + \tan x) \quad \text{Ans.}$$

Q.18 $(e^x + e^{-x}) \frac{dy}{dx} = e^x - e^{-x} \quad (\text{Lhr. Board 2011})$

Solution:

$$(e^x + e^{-x}) \frac{dy}{dx} = e^x - e^{-x}$$

Separate variables

$$dy = \frac{e^x - e^{-x}}{e^x + e^{-x}} \, dx$$

Integrate

$$\int dy = \int \frac{e^x - e^{-x}}{e^x + e^{-x}} \, dx$$

$$y = \ln(e^x + e^{-x}) + c \quad \text{Ans.}$$

Q.19 Find the general solution of the equation $\frac{dy}{dx} - x = xy^2$. Also find the particular solution if $y = 1$ when $x = 0$.

Solution:

$$\frac{dy}{dx} - x = xy^2$$

To find General Solution



$$\frac{dy}{dx} - x = xy^2$$

Separate variables

$$\frac{dy}{dx} = xy^2 + x$$

$$\frac{dy}{dx} = x(y^2 + 1)$$

$$\frac{dy}{y^2 + 1} = x dx$$

Integrate

$$\int \frac{dy}{y^2 + 1} = \int x dx$$

$$\tan^{-1} y = \frac{x^2}{2} + c \quad \text{———— (1)}$$

To find particular solution

Put $y = 1$, $x = 0$ in equation (1)

$$\tan^{-1}(1) = \frac{(0)^2}{2} + c$$

$$c = \frac{\pi}{4}$$

Put $c = \frac{\pi}{4}$ in equation (1)

$$\tan^{-1} y = \frac{x^2}{2} + \frac{\pi}{4} \quad \text{Ans.}$$

Q.20 Solve the differential equation $\frac{dx}{dt} = 2x$ given $x = 4$ when $t = 0$.

Solution:

$$\frac{dx}{dt} = 2x$$

Separate variable

$$\frac{dx}{x} = 2dt$$

Integrate

$$\int \frac{dx}{x} = 2 \int dt$$

$$\ln x = 2t + \ln c$$

$$e^{\ln x} = e^{2t + \ln c}$$



$$\begin{aligned}
 e^{\ln x} &= e^{2t} \cdot e^{\ln c} \\
 x &= ce^{2t} \quad \text{————— (1)} \\
 \text{Put } x &= 4, \quad t = 0 \text{ in equation (1)} \\
 4 &= ce^{2(0)} \\
 c &= 4 \\
 \text{Put } c &= 4 \text{ in equation (1)} \\
 x &= 4e^{2t} \quad \text{Ans.}
 \end{aligned}$$

Q.21 Solve the differential equation $\frac{ds}{dt} + 2st = 0$. Also find the particular solution if $s = 4e$, when $t = 0$.

Solution:

$$\frac{ds}{dt} + 2st = 0$$

Separate variables

$$\frac{ds}{dt} = -2st$$

$$\frac{ds}{s} = -2tdt$$

Integrate

$$\int \frac{ds}{s} = -2 \int t dt$$

$$\ln s = -\frac{2t^2}{2} + \ln c$$

$$e^{\ln s} = e^{-t^2} + \ln c$$

$$s = e^{-t^2} \cdot e^{\ln c}$$

$$\boxed{s = ce^{-t^2}} \quad \text{————— (1)}$$

To find particular solution

$$\text{Put } s = 4e, \quad t = 0 \text{ in equation (1)}$$

$$4e = ce^0$$

$$c = 4e$$

$$\text{Put } c = 4e \text{ in equation (1)}$$

$$s = 4e \cdot e^{-t^2}$$

$$\boxed{s = 4e^{1-t^2}} \quad \text{Ans.}$$



Q.22 In a culture bacteria increases at the rate proportional to the number of bacteria present. If bacteria are 200 initially and are doubled in 2 hours, find the number of bacteria present four hours later.

Solution:

Let P be the number of bacteria present at time t .

$$\frac{dp}{dt} = kP \quad (k > 0)$$

Separate variable

$$\frac{dp}{p} = k dt$$

Integrate

$$\int \frac{dp}{p} = k \int dt$$

$$\ln p = kt + c_1 \quad \text{————— (1)}$$

Put $p = 200$, $t = 0$ in equation (1)

$$\ln 200 = 0 + c_1$$

$$c_1 = \ln 200$$

Put $c_1 = \ln 200$ in equation (1)

$$\ln p = kt + \ln 200 \quad \text{————— (2)}$$

Put $P = 400$, $t = 2$ hour in equation (2)

$$\ln 400 = 2k + \ln 200$$

$$\ln 400 - \ln 200 = 2k$$

$$k = \frac{1}{2} \ln \left(\frac{400}{200} \right)$$

$$k = \frac{1}{2} \ln 2$$

Put $k = \frac{1}{2} \ln 2$ in equation (2)

$$\ln P = \frac{1}{2} \ln 2t + \ln 200 \quad \text{————— (3)}$$

To find the number of bacteria presents four hour later

Put $t = 4$ hour in equation (3)

$$\ln P = \frac{1}{2} \ln 2 \times 4 + \ln 200$$

$$\ln P = 2 \ln 2 + \ln 200$$

$$\ln P = \ln 2^2 + \ln 200$$

$$\ln P = \ln (4 \times 200)$$

$$\boxed{P = 800} \quad \text{Ans.}$$



Q.23 A ball is thrown vertically upward with a velocity of 2450 cm/sec. Neglecting air resistance, find

- (i) velocity of ball at any time t . (ii) distance travelled in any time t .
 (iii) maximum height attained by the ball.

Solution:

- (i) Let 'v' be the velocity at any time 't' then by Newton's law of motion

$$\frac{dv}{dt} = -g$$

Separate variables

$$dv = -g dt$$

Integrate $\int dv = -g \int dt$

$$v = -gt + c_1 \quad (1)$$

Put $v = 2450$ cm/sec and $t = 0$ in eq. (1)

$$2450 = 0 + c_1$$

$$c_1 = 2450$$

Put $c_1 = 2450$ in equation (1)

$$v = -980t + 2450 \quad (2) \quad (\text{taking } g = 980)$$

- (ii) Let 'h' be the height attained at any time 't'.

$$\frac{dh}{dt} = 2450 - 980t$$

Separate variables

$$dh = 2450 dt - 980t dt$$

Integrate $\int dh = 2450 \int dt - 980 \int t dt$

$$h = 2450t - 980 \frac{t^2}{2} + c_2$$

$$h = 2450t - 490t^2 + c_2 \quad (3)$$

Put $h = 0$, $t = 0$ in equation (3)

$$0 = 0 - 0 + c_2$$

$$c_2 = 0$$

Put $c_2 = 0$ in equation (3)

$$h = 2450t - 490t^2 \quad (4)$$

- (iii) The maximum height attained by the ball when $v = 0$

$$-980t + 2450 = 0$$

$$980t = 2450$$

$$t = \frac{2450}{980} = 2.5 \text{ Sec}$$

Put $t = 2.5$ sec in equation (4)

$$h = 2450(2.5) - 490(2.5)^2$$

$$h = 6125 - 490(6.25) = 6125 - 3062.5$$

$$h = 3062.5 \text{ cm} \quad \text{Ans.}$$

